Yoneda structures from 2-toposes

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Abstract. A 2-categorical generalisation of the notion of elementary topos is provided, and some of the properties of the yoneda structure [**SW78**] it generates are explored. Results enabling one to exhibit objects as cocomplete in the sense definable within a yoneda structure are presented. Examples relevant to the globular approach to higher dimensional category theory are discussed. This paper also contains some expository material on the theory of fibrations internal to a finitely complete 2-category [**Str74b**] and provides a self-contained development of the necessary background material on yoneda structures.

1. Introduction

The idea of an internal category is due to Ehresmann [**Ehr58**]. This notion has experienced somewhat of a resurgence in recent times because of developments in higher category theory. For example in the work of Michael Batanin [**Bat98b**] [**Bat98a**] [**Bat02**] [**Bat03**] internal category theory is the backdrop for the theory of higher operads. In the work of John Baez and his collaborators [**BL04**] [**BC04**] [**BS05**] we see internal categories as fundamental in the process of categorifying differential geometry and gauge theory with a view to applications in physics.

This paper is about doing category theory internally with a particular focus on the theory of colimits. It was motivated by the need to manipulate internal colimits more easily in order to push forward the theory of higher operads. In [**Web**] the results and notions of the present paper will be used to bring all the operad theory of [**Bat98b**] to the level of generality of [**Web05**] so that the theory of higher symmetric operads is facilitated. This will then be used in future work on reconciling the notions of higher dimensional categories in [**BD98**] and [**Bat98b**], and in an operadic exploration of the stabilisation hypothesis [**BD95**].

In the 1970's internal colimits were understood from a 2-categorical perspective in the work of Ross Street and Bob Walters [**SW78**]. In that paper the concept of a *yoneda structure* on a 2-category was discovered; inspired largely by the work of Bill Lawvere on the foundational importance of the category of categories [**Law70**]. Logical motivations notwithstanding, the perspective of this paper is that the point of having a yoneda structure on a 2-category $\mathcal{K}$, is that one can then say what it means for an object of $\mathcal{K}$ to be *cocomplete* in such a way that the theory of cocompleteness develops in $\mathcal{K}$ as in ordinary category theory. The resulting theory of Street and Walters clarifies colimits in both enriched and internal category theory, and is surprisingly simple.

In this paper we focus attention on the yoneda structures that arise in internal category theory. These yoneda structures satisfy some additional properties described in definition(3.1). We recall the resulting theory of colimits that arises in this setting in section(3).

The examples of interest for us involve a 2-category $\mathcal{K}$ equipped with an object $\Omega \in \mathcal{K}$ which plays the role of an internal category of sets for $\mathcal{K}$. In the paradigmatic example we consider the category Set of small sets, another category SET of sets which contains the arrows of Set as an object, and CAT as the 2-category of categories internal to SET. For this example $\mathcal{K}$ is CAT and Ω is Set. In higher category theory one obtains another important example by taking $\mathcal{K}$ to be the 2-category of globular categories and Ω to be the globular category of higher spans of sets [**Bat98b**] [**Str00**].

Yoneda structures arise from the setting alluded to in the previous paragraph because Ω satisfies a property analogous to that enjoyed by the subobject classifier of an elementary topos. That this is so in the paradigmatic example described above is an important observation of Bill Lawvere. A 2-categorical expression of this property is provided in the work of Ross Street by the notion of a *fibrational cosmos* [**Str74a**] [**Str80a**]. The *2-toposes* of this work are simply cosmoses whose underlying 2-category is cartesian closed and comes equipped with a duality involution[1]. This notion is isolated here because it is easier to exhibit examples of 2-toposes and to explore their properties. We exploit this to understand better some of the yoneda structures that arise. In particular, part of any yoneda structure is a presheaf construction: an assignment of an object $\widehat{A}$ for any *admissible* object $A \in \mathcal{K}$ to be regarded as the object of presheaves on A. The main results of this paper describe when the yoneda structures that arise from 2-toposes have presheaf objects which are cocomplete.

Our basic references for background on 2-categories is [**KS74**] and [**Str80b**]. Another important background article is [**Str74b**] although efforts are made in this paper to keep the exposition relatively self-contained. The 2-categorical background pertinent to this work is collected in section(2), and the definition of a duality involution is provided in subsection(2.3).

In section(4) the notion of 2-topos is defined and the basic examples are presented, and in section(5) the yoneda structure arising from a 2-topos is described. Not all 2-toposes give yoneda structures with cocomplete presheaf objects as we see in example(6.4). In fact this example and example(8.3) give some idea of the things that cannot be taken for granted in an arbitrary 2-topos. Part of the axiomatics of a yoneda structure is a right ideal of arrows called *admissible maps*. Section(6) also provides a basic result to help characterise the admissible maps in some of our examples. Section(7) develops the results on presheaf cocompleteness. In section(8) we exhibit Ω as a cartesian closed object of $\mathcal{K}$ under some hypotheses on the 2-topos $\mathcal{K}$ (which include the cocompleteness of Ω). Applied to the globular category of higher spans, the results of this paper say that this globular category is the small globular colimit completion of 1, and that it is cartesian closed as a globular category.

[1]Less general notions of 2-toposes were considered in [**Pen74**] and [**Bou74**], in which Ω is used to classify cosieves (see example(4.3)) for a cartesian closed finitely complete 2-category $\mathcal{K}$.

In all the examples considered in this work $\mathcal{K}$ is a 2-category of categories internal to some nice category, and so one might wonder why bother with a 2-categorical abstraction? One reason for this is that the theory just comes out easier when things are expressed this way. Lax and pseudo pullbacks are very useful things. However the main reason is that it is 2-toposes *together* with nice 2-monads on them which provide a conceptual basis for the theory of operads [**Web**], and many of these nice 2-monads do not arise from nice monads on the categories in which we internalise[2]. This is especially so when one wishes to study weakly symmetric higher dimensional monoidal categories.

Consistent with the notation for a yoneda structure, for a category $\mathbb{C}$ we denote by $\widehat{\mathbb{C}}$ the category of presheaves on $\mathbb{C}$, that is the functor category $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}]$. When working with presheaves we adopt the standard practises of writing C for the representable $\mathbb{C}(-, C) \in \widehat{\mathbb{C}}$ and of not differentiating between an element $x \in X(C)$ and the corresponding map $x : C \to X$ in $\widehat{\mathbb{C}}$. We denote by $\mathrm{CAT}(\widehat{\mathbb{C}})$ the functor 2-category $[\mathbb{C}^{\mathrm{op}}, \mathrm{CAT}]$ which consists of functors $\mathbb{C}^{\mathrm{op}} \to \mathrm{CAT}$, natural transformations between them and modifications between those. We adopt the standard notations for the various duals of a 2-category $\mathcal{K}$: $\mathcal{K}^{\mathrm{op}}$ is obtained from $\mathcal{K}$ by reversing just the 1-cells, $\mathcal{K}^{\mathrm{co}}$ is obtained by reversing just the 2-cells, and $\mathcal{K}^{\mathrm{coop}}$ is obtained by reversing both the 1-cells and the 2-cells.

2. 2-categorical preliminaries

2.1. Finitely complete 2-categories. Recall that a 2-category $\mathcal{K}$ is *finitely complete* when it admits all limits weighted by 2-functors $I : J \to \mathrm{CAT}$, in the sense of CAT-enriched category theory [**Kel82**], such that the set of 2-cells of J and the sets of arrows of the $I(j)$ for $j \in J$, are all finite. This means that for such I and $T : J \to \mathcal{K}$ there is an object $\lim(I, T)$ of $\mathcal{K}$ and isomorphisms of categories

$$\mathcal{K}(X, \lim(I, T)) \cong [J, \mathrm{CAT}](I, \mathcal{K}(X, T))$$

2-natural in X[3]. From [**Str76**] one has the result that $\mathcal{K}$ is finitely complete iff it admits finite conical limits and cotensors with the ordinal 2. Slightly less efficiently $\mathcal{K}$ is finitely complete iff it has a terminal object, pullbacks and lax pullbacks[4] (also known as comma objects). Thus the discussion of finitely complete 2-categories differs from the discussion of finitely complete categories only because for 2-categories there are different types of pullback: one can consider the lax pullback, pseudo pullback or pullback of a pair of arrows f and g depending on whether one is considering squares

$$\begin{array}{ccc} P & \longrightarrow & A \\ \big\downarrow & \overset{\phi}{\Rightarrow} & \big\downarrow g \\ B & \underset{f}{\longrightarrow} & C \end{array}$$

[2]The most basic example of this is the 2-monad on CAT whose (strict) algebras are symmetric (strict) monoidal categories.

[3]However as explained in [**Str76**] finitely complete 2-categories in the sense recalled here admit limits weighted by a more general class of weight. That is, there are weights I whose limits exist in any finitely complete 2-category but are such that the categories $I(j)$ are *not* finite.

[4]Note that the terms *lax pullback* and *pseudo pullback* have been used in a slightly different way in other work.

in $\mathcal{K}$ such that ϕ is a general 2-cell, invertible or an identity respectively. In addition to these variations one can also weaken the universal property and consider bicategorical limits [**Str80b**] but in this paper we shall only consider the stronger notion.

Recall that the *lax pullback* of

$$A \xrightarrow{f} B \xleftarrow{g} C$$

in a 2-category $\mathcal{K}$ consists of

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

in $\mathcal{K}$ universal in the following sense:

(1) given

$$\begin{array}{ccc} X & \xrightarrow{k} & C \\ {\scriptstyle h}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

there is a unique $\delta : X \to f/g$ such that $p\delta = h$, $q\delta = k$ and $\lambda\delta = \phi$.

(2) for δ_1 and $\delta_2 : X \to f/g$, given $\alpha : p\delta_1 \to p\delta_2$ and $\gamma : q\delta_1 \to q\delta_2$ such that

$$\begin{array}{ccc} fp\delta_1 & \xrightarrow{\lambda\delta_1} & gq\delta_1 \\ {\scriptstyle f\alpha}\downarrow & = & \downarrow{\scriptstyle g\beta} \\ fp\delta_2 & \xrightarrow[\lambda\delta_2]{} & gq\delta_2 \end{array}$$

there is a unique $\pi : \delta_1 \to \delta_2$ such that $p\pi = \alpha$ and $q\pi = \gamma$.

It is standard notation when fixing a choice of lax pullback to write f/g in the place of P. The *pseudo pullback* $f/_{\cong}g$ in $\mathcal{K}$ is defined in the same way except that the 2-cells ϕ and λ in the above definition are invertible, and the *pullback* as $f/_{=}g$ may be defined in the same way as above except that ϕ and λ are identities.

Example 2.1. Lax pullbacks in CAT are very well known. Given functors

$$A \xrightarrow{f} B \xleftarrow{g} C$$

one can define the category f/g as follows:

- objects are triples $(a, h : fa \to gc, c)$ where $a \in A$, $c \in C$, and $h \in B$.
- an arrow $(a, h, c) \to (a', h', c')$ is a pair $(\alpha : a \to a', \gamma : c \to c')$ such that $h'f(\alpha) = g(\gamma)h$.
- composition and identities induced from the category structures of A, B and C.

and one has

$$\begin{array}{ccc} f/g & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

where $p(a,h,c) = a$, $q(a,h,c) = c$ and $\lambda_{(a,h,c)} = h$, satisfying the universal property of a lax pullback. One obtains $f/_{\cong}g$ as the full subcategory of f/g consisting of the (a,h,c) such that h is invertible, and $f/_{=}g$ as the full subcategory of f/g consisting of the (a,h,c) such that h is an identity arrow. Conversely given lax pullbacks in CAT one can define general lax pullbacks as follows. A square

$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

in a 2-category $\mathcal{K}$ exhibits P as f/g iff for all $X \in \mathcal{K}$ the functor

$$\mathcal{K}(X,P) \to \mathcal{K}(X,f)/\mathcal{K}(X,g)$$

induced by $\mathcal{K}(X,\lambda)$ is an isomorphism of categories. This is called the *representable* definition of lax pullbacks in $\mathcal{K}$. Similarly, pseudo pullbacks, pullbacks and indeed all 2-categorical limits can be defined representably.

EXAMPLE 2.2. Let $\mathcal{K}$ be a 2-category and $A \in \mathcal{K}$. It is standard to denote by $\mathcal{K}/A$ the 2-category formed as the lax pullback of

$$\mathcal{K} \xrightarrow{1_{\mathcal{K}}} \mathcal{K} \xleftarrow{A} 1$$

in the 2-category of 2-categories, 2-functors and 2-natural transformations. Its explicit description is similar to its categorical analogue in that the objects of $\mathcal{K}/A$ are arrows $f : X \to A$, and a morphism $f_1 \to f_2$ is an arrow g of $\mathcal{K}$ such that $f_2 g = f_1$. However a 2-cell $\gamma : g_1 \Rightarrow g_2$ of $\mathcal{K}/A$ is a 2-cell $\gamma : g_1 \Rightarrow g_2$ of $\mathcal{K}$ such that $f_2\gamma$ is an identity. Recall [**CJ95**] that the 2-functor $\mathcal{K}/A \to \mathcal{K}$ whose object map takes the domain of an arrow into A, creates any connected limits that exist in $\mathcal{K}$. When $\mathcal{K}$ is finitely complete and $f : A \to B$ in $\mathcal{K}$, the processes of taking the pullback, pseudo pullback and lax pullback along f provide 2-functors $\mathcal{K}/B \to \mathcal{K}/A$.

EXAMPLE 2.3. Let $\mathbb{C}$ be a category. Then a lax pullback f/g of

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xleftarrow{g} & C \\ & {\scriptstyle \alpha}\searrow = & \downarrow{\scriptstyle \beta} & = \swarrow{\scriptstyle \gamma} & \\ & & \mathbb{C} & & \end{array}$$

in CAT/$\mathbb{C}$ can be specified as follows: the domain category is the full subcategory of f/g as defined in CAT as in the previous example, consisting of the (a,h,c) such that $\beta(h)$ is an identity; and the functor into $\mathbb{C}$ sends such (a,h,c) to $\alpha(a) = \gamma(c)$ in $\mathbb{C}$. Notice that while the domain 2-functor CAT/$\mathbb{C} \to$CAT preserves pullbacks, from the description of lax pullbacks in CAT/$\mathbb{C}$ given here, it does *not* preserve lax pullbacks in general.

Example 2.4. Let $\mathcal{E}$ be a category with pullbacks. Recall the 2-category $\mathrm{Cat}(\mathcal{E})$ of categories internal to $\mathcal{E}$. For $A \in \mathrm{Cat}(\mathcal{E})$ it is standard to denote by

$$A_0 \underset{t}{\overset{s}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i}}}}} A_1 \;\underset{\longleftarrow}{\overset{\longleftarrow}{\xleftarrow{c}}}\!\!\!\!\!\!\rightrightarrows A_2$$

the 2-truncated simplicial diagram determined by A: A_0 is the object of objects of A, A_1 the object of arrows, s the source or domain map, t the target or codomain map, i the map that provides identity arrows, A_2 the object of composable pairs in A obtained by pulling back s along t, and c the composition map. Using pullbacks only pullbacks in $\mathcal{E}$ one can construct pullbacks, pseudo pullbacks and lax pullbacks in $\mathrm{Cat}(\mathcal{E})$. One can either do this directly, or by interpretting the explicit description of these constructions in CAT in the internal language of $\mathcal{E}$ (see [**Joh02**]). A nice consequence of this is that for any pullback preserving functor $\mathcal{E}{\to}\mathcal{E}'$, the 2-functor $\mathrm{Cat}(\mathcal{E}){\to}\mathrm{Cat}(\mathcal{E}')$ it induces preserves pullbacks, pseudo pullbacks and lax pullbacks.

Lax pullbacks satisfy the same composition and cancellation properties as pullbacks do in ordinary category theory. That is, given

$$\begin{array}{ccccc} X & \longrightarrow & A & \xrightarrow{v} & B \\ \downarrow & = & \downarrow{\scriptstyle u} & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[h]{} & C & \xrightarrow[f]{} & D \end{array}$$

such that λ exhibits A as f/g, then the composite square exhibits X as fh/g iff the front square is a pullback. Similarly for pseudo pullbacks and pullbacks in any 2-category. So for example, one can use this observation to obtain all lax pullbacks from pullbacks and lax pullbacks of identity arrows. We adopt some standard notational abuses with regards to identity arrows: $f/1_A$ is written f/A, $1_A/g$ is written A/g and $1_A/1_A$ is written A/A.

Some other notation: we shall denote a terminal object of $\mathcal{K}$ by 1 and for $A \in \mathcal{K}$ the unique map $A{\to}1$ by t_A.

2.2. Fibrations. The concept of a fibration between categories is due to Grothendieck [**Gro70**]. Fibrations can be defined internal to any finitely complete 2-category $\mathcal{K}$. In fact there are two approaches: one can work 2-categorically and follow [**Str74b**], or one can regard $\mathcal{K}$ as a bicategory following [**Str80b**]. For the case $\mathcal{K} = \mathrm{CAT}$ the 2-categorical definition of fibration coincides with that of Grothendieck [**Gra66**] whereas the bicategorical definition of fibration is more general. In this paper we shall consider only the stronger notion.

Let $f : A{\to}B$ be a functor. A morphism $\alpha : a_1{\to}a_2$ in A is *f-cartesian* when for all α_1 and β as shown:

$$\begin{array}{ccc} fa_1 & \xrightarrow{f\alpha} & fa_2 \\ {\scriptstyle \beta}\uparrow & = & \nearrow{\scriptstyle f\alpha_1} \\ fa_3 & & \end{array}$$

there is a unique $\gamma : a_3{\to}a_1$ such that $f\gamma = \beta$ and $\alpha\gamma = \alpha_1$. The basic facts about f-cartesian morphisms are:

(1) if $a_1 \xrightarrow{\alpha_1} a_2 \xrightarrow{\alpha_2} a_3$ are in A and α_2 is f-cartesian, then α_1 is f-cartesian iff $\alpha_2\alpha_1$ is f-cartesian.

(2) isomorphisms are f-cartesian for any f.
(3) if α is f-cartesian and $f\alpha$ is an isomorphism then α is an isomorphism.

A *cartesian lift* of the pair $(\beta : b \to fa, a)$ is an f-cartesian morphism $\alpha : a_2 \to a$ such that $f\alpha = \beta$. Grothendieck defined f to be a *fibration* when every $(\beta : b \to fa, a)$ has a cartesian lift.

Let $\mathcal{K}$ be a finitely complete 2-category and $f : A \to B$ be in $\mathcal{K}$. A 2-cell

$$X \underset{a_2}{\overset{a_1}{\rightrightarrows}} A \qquad \Downarrow \alpha$$

is *f-cartesian* when for all $g : Y \to X$ in $\mathcal{K}$, $\alpha g : \alpha a_1 \to \alpha a_2$ is a $\mathcal{K}(Y,f)$-cartesian morphism in $\mathcal{K}(Y,A)$. One then defines f to be a *fibration* when for all

$$\begin{array}{ccc} X & \xrightarrow{\;a\;} & A \\ & {\scriptstyle b}\searrow \;\overset{\beta}{\Rightarrow}\; \swarrow{\scriptstyle f} & \\ & B & \end{array}$$

there exists f-cartesian $\overline{\beta} : c \Rightarrow a$ so that $fc = b$ and $f\overline{\beta} = \beta$. The following well known result is fundamental and easy to prove.

THEOREM 2.5. *In any finitely complete 2-category $\mathcal{K}$:*
(1) *The composite of fibrations is a fibration.*
(2) *The pullback of a fibration along any map is a fibration.*

It is natural to consider applying the cartesian lifting criterion of a fibration f to the 2-cell from a lax pullback involving f. When one does this, one is lead to theorem(2.7). Given $f : A \to B$ and $g : X \to B$ in a finitely complete 2-category $\mathcal{K}$, we shall denote by $i : g/_{=}f \to g/f$ the map induced by the universal property of g/f and the identity cell

$$\begin{array}{ccc} g/_{=}f & \longrightarrow & A \\ \downarrow & \overset{\text{id}}{\Rightarrow} & \downarrow {\scriptstyle f} \\ X & \xrightarrow[g]{} & B \end{array}$$

of a defining pullback square for $g/_{=}f$.

LEMMA 2.6. *Let $\mathcal{K}$ be a 2-category and $i : X \to Y$ in $\mathcal{K}$. Then to give $i \dashv r$ with invertible unit, it suffices to give $\varepsilon : ir \Rightarrow 1_Y$ such that $r\varepsilon$ and εi are invertible and $1_X \cong ri$.*

PROOF. Write $\eta : 1_X \to ri$ for the isomorphism. In general η and ε won't necessarily satisfy the triangular equations of an adjunction, so we define $\eta' : 1_X \to ri$ to be the composite

$$1 \xrightarrow{\eta} ri \xrightarrow{(r\varepsilon i)^{-1}} riri \xrightarrow{(ri\eta)^{-1}} ri$$

and we must verify that $\eta' r = (r\varepsilon)^{-1}$ and $i\eta' = (\varepsilon i)^{-1}$. Observe that

$$\begin{array}{ccc} 1 & \xrightarrow{\eta} & ri \\ {\scriptstyle \eta}\downarrow & = & \downarrow{\scriptstyle \eta ri} \\ ri & \xrightarrow[ri\eta]{} & riri \end{array} \qquad \begin{array}{ccc} ririr & \xrightarrow{r\varepsilon ir} & rir \\ {\scriptstyle rir\varepsilon}\downarrow & = & \downarrow{\scriptstyle r\varepsilon} \\ rir & \xrightarrow[r\varepsilon]{} & r \end{array} \qquad \begin{array}{ccc} irir & \xrightarrow{\varepsilon iri} & iri \\ {\scriptstyle ir\varepsilon i}\downarrow & = & \downarrow{\scriptstyle \varepsilon i} \\ iri & \xrightarrow[\varepsilon i]{} & i \end{array}$$

and so $ri\eta = \eta ri$, $rir\varepsilon = r\varepsilon ir$ and $ir\varepsilon i = \varepsilon iri$. From these equations, the definition of η' and

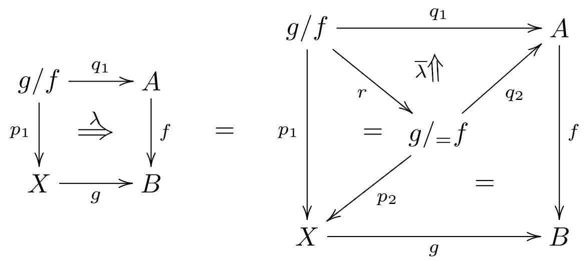
$$\begin{array}{ccc} rir & \xrightarrow{\eta rir} & ririr \\ {\scriptstyle r\varepsilon}\downarrow & = & \downarrow{\scriptstyle rir\varepsilon} \\ r & \xrightarrow[\eta r]{} & rir \end{array} \qquad \begin{array}{ccc} iri & \xrightarrow{iri\eta} & iriri \\ {\scriptstyle \varepsilon i}\downarrow & = & \downarrow{\scriptstyle \varepsilon iri} \\ i & \xrightarrow[i\eta]{} & iri \end{array}$$

the equations: $\eta' r = (r\varepsilon)^{-1}$ and $i\eta' = (\varepsilon i)^{-1}$ follow immediately. □

Theorem 2.7. [**Str74b**] *Let $\mathcal{K}$ be a finitely complete 2-category and $f : A \to B$ in $\mathcal{K}$. Then the following statements are equivalent:*

1. *f is a fibration.*
2. *For all $g : X \to B$, the map $i : g/_{=}f \to g/f$ has a right adjoint in $\mathcal{K}/X$.*
3. *The map $i : A \to B/f$ has a right adjoint in $\mathcal{K}/B$.*

Proof. (1)⇒(2): Apply the above definition to $\phi = \lambda$ the defining lax pullback for g/f, to obtain

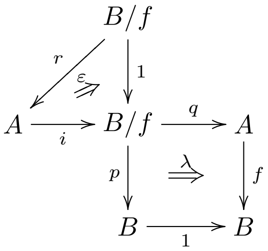


where $\overline{\lambda}$ is f-cartesian, and so by the universal property of g/f we obtain $\varepsilon : ir \Rightarrow 1$ such that $p_1\varepsilon = \mathrm{id}$ and $q_1\varepsilon = \overline{\lambda}$. By lemma(2.6) it suffices to show that εi and $r\varepsilon$ are invertible, and that $ri \cong 1$ in $\mathcal{K}/X$. Observe that $q\varepsilon i = \overline{\lambda} i$ is an f-cartesian lift of an identity cell, and thus is invertible. Since $p\varepsilon i = \mathrm{id}$, the 2-cell εi is invertible by the universal property of g/f. So we have $\overline{\lambda} i : q_2 ri \cong q_2$ and $p_2 ri = p_2$, and by the above defining diagram/equation of $\overline{\lambda}$ precomposed with i, together with the universal property of $g/_{=}f$, one obtains $\eta : ri \cong 1$ such that $p\eta = \mathrm{id}$. Finally to see that $r\varepsilon$ is invertible, by the universal property of $g/_{=}f$, it suffices to show that $q_2 r\varepsilon$ is invertible, since $p_2 r\varepsilon = p_1\varepsilon = \mathrm{id}$. Note that $fq_2 r\varepsilon = gp_2 r\varepsilon = gp_1\varepsilon = \mathrm{id}$, and so it suffices to show that $q_2 r\varepsilon$ is f-cartesian. For this we note that

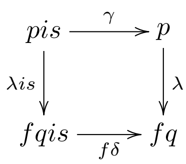
$$\begin{array}{ccc} q_2 rir & \xrightarrow{q_2 r\varepsilon} & q_2 r \\ {\scriptstyle \overline{\lambda} i_r}\downarrow & & \downarrow{\scriptstyle \overline{\lambda}} \\ q_1 ir & \xrightarrow[q_1\varepsilon]{} & q_1 \end{array}$$

commutes, and that since $\overline{\lambda} = q_1\varepsilon$ is f-cartesian, $q_2 r\varepsilon$ is f-cartesian also.
(2)⇒(3): just take $g = 1_B$.
(3)⇒(1): By the universal property of B/f it suffices to verify the defining property of a fibration as defined above for the case $\beta = \lambda$, the defining lax pullback cell for

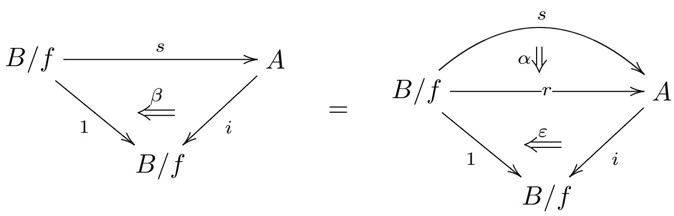
B/f. We have



where ε is the counit for the adjunction $i \dashv r$ in $\mathcal{K}/B$. Since $\lambda i = \mathrm{id}$ this composite evaluates to $fq\varepsilon$, and since $p\varepsilon = \mathrm{id}$ this composite also evaluates to λ whence $\lambda = fq\varepsilon$. Thus it suffices to show that $q\varepsilon$ is f-cartesian. To this end let $\delta : s \to q$ and $\gamma : fs \to fr$ such that $f\delta = f(q\varepsilon)\gamma$. Another way to express this last equation, since $\lambda = fq\varepsilon$ and $\lambda is = \mathrm{id}$, is that



commutes, and so by the universal property of B/f, there is a unique $\beta : is \to 1$ such that $p\beta = \gamma$ and $q\beta = \delta$. Since ε exhibits r as a right lifting of 1 along i (see example(2.17) below), there is a unique α such that



Post composing this last equation with p gives $\gamma = f\alpha$, and post composing it with q gives $\delta = (q\varepsilon)\alpha$. Conversely α is clearly the unique 2-cell satisfying these equations. ☐

A functor $p : E \to B$ is an *opfibration* when the functor $p^{\mathrm{op}} : E^{\mathrm{op}} \to B^{\mathrm{op}}$ is a fibration. These functors were originally called cofibrations by Grothendieck, and indeed they are fibrations in $\mathrm{CAT}^{\mathrm{co}}$, however beginning with Gray [**Gra66**] the term opfibration was used instead so as not to give topologists the wrong idea: it is the fibrations in $\mathrm{CAT}^{\mathrm{op}}$ which are more like what a topologist would call a cofibration, and both fibrations and opfibrations with their lifting properties which correspond intuitively to topologists' fibrations. Similarly one refers to *opcartesian liftings*, defines opfibrations in an arbitrary 2-category $\mathcal{K}$ as fibrations in $\mathcal{K}^{\mathrm{co}}$, and interprets the above results in $\mathcal{K}^{\mathrm{co}}$ when working with this dual notion. By the characterisation of fibrations given in theorem(2.7)(3) and its dual one has the following immediate corollary.

Corollary 2.8. *If a 2-functor between finitely complete 2-categories preserves lax pullbacks, then it preserves fibrations and opfibrations.*

Example 2.9. In this example we revisit fibrations in CAT from the present general point of view and recall the Grothendieck construction for later use. To see that fibrations in CAT are indeed Grothendieck fibrations let $f : A \to B$ be a functor. To give a right adjoint to $i : A \to B/f$ over B, one must give for each pair $(\beta : b \to fa, a)$ an object $r(\beta, a)$ in A such that $fr(\beta, a) = b$, and an arrow $\varepsilon_{(\beta,a)} : r(\beta, a) \to a$ such that $f\varepsilon_{\beta,a} = \beta$ and so that the map

$$(\mathrm{id}, \varepsilon_{(\beta,a)}) : (\mathrm{id}, r(\beta, a)) \to (\beta, a)$$

in B/f has the universal property of the counit of an adjunction. This universal property amounts to $\varepsilon_{(\beta,a)}$ being f-cartesian. Thus a right adjoint to $i : A \to B/f$ over B amounts to a choice of cartesian liftings for f. A *cleavage* is the standard terminology for such a choice of cartesian liftings, and a fibration together with a cleavage is known as a *cloven* fibration. Given a pseudo functor $X : \mathbb{C}^{\mathrm{op}} \to \mathrm{CAT}$ the Grothendieck construction produces an associated cloven fibration over $\mathbb{C}$. Define the category $\mathrm{el}(X)$ as follows:

- objects: are pairs (x, C) where $C \in \mathbb{C}$ and $x \in X(C)$.
- an arrow $(x_1, C_1) \to (x_2, C_2)$ is a pair (α, β) where $\beta : C_1 \to C_2$ in $\mathbb{C}$ and $\alpha : x_1 \to X\beta(x_2)$.
- compositions: formed in the evident way using the pseudo functor coherence cells and composition in $\mathbb{C}$.

define the functor into $\mathbb{C}$ to be the obvious forgetful functor, and note that this functor has an obvious cleavage. This construction provides a 2-equivalence $\mathrm{Fib}(\mathbb{C}) \simeq \mathrm{Hom}(\mathbb{C}^{\mathrm{op}}, \mathrm{CAT})$[5]. A useful perspective on this last fact is that the 2-functor

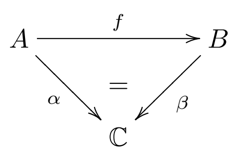
$$\mathrm{el} : \mathrm{CAT}(\widehat{\mathbb{C}}) \to \mathrm{CAT}$$

factors as

$$\mathrm{CAT}(\widehat{\mathbb{C}}) \longrightarrow \mathrm{CAT}/\mathbb{C} \longrightarrow \mathrm{CAT}$$

where the first 2-functor is monadic and the second takes the domain of a functor into $\mathbb{C}$. The induced 2-monad on on $\mathrm{CAT}/\mathbb{C}$ has functor part given by taking lax pullbacks along $1_{\mathbb{C}}$ and the 2-category of pseudo algebras for this 2-monad is 2-equivalent to $\mathrm{Fib}(\mathbb{C})$. The general idea of seeing fibrations as algebras of a 2-monad in this way was developed in [**Str74b**]. Note also that this factorisation of el implies that el preserves all connected conical limits [**CJ95**].

Example 2.10. For a category $\mathbb{C}$ we shall now consider fibrations in $\mathrm{CAT}/\mathbb{C}$. In example(2.3) we described explicitly the construction of lax pullbacks in $\mathrm{CAT}/\mathbb{C}$. Thus one can unpack the definition of fibration in $\mathrm{CAT}/\mathbb{C}$ in much the same way as we did for CAT in the previous example. When one does this one finds that



is a fibration in $\mathrm{CAT}/\mathbb{C}$ iff every $(\phi : b \to fa, a)$ such that $\beta(\phi) = \mathrm{id}$ has a cartesian lift. In this way the idea that f be "locally" a fibration, or in other words a

[5] $\mathrm{Hom}(\mathbb{C}^{\mathrm{op}}, \mathrm{CAT})$ is the 2-category of pseudo functors $\mathbb{C}^{\mathrm{op}} \to \mathrm{CAT}$, pseudo natural tranformations between them, and modifications between those and $\mathrm{Fib}(\mathbb{C})$ is the sub-2-category of $\mathrm{CAT}/\mathbb{C}$ consisting of fibrations and functors over $\mathbb{C}$ that preserve cartesian arrows.

"fibration on the fibres of β", is formalised by saying that f is a fibration in CAT/$\mathbb{C}$. In particular notice that the condition that f be a fibration over $\mathbb{C}$ is in general weaker than the condition that f be a fibration, although they are equivalent when f lives in Fib($\mathbb{C}$) [**Bén85**] [**Her99**] as we shall now recall. For such an f let $\phi : b \to fa$. To obtain a cartesian lift for ϕ one first takes a cartesian lift $\phi_1 : a_1 \to a$ of $\beta(\phi)$. Since f preserves cartesian arrows ϕ factors uniquely as

$$b \xrightarrow{\phi_2} fa_1 \xrightarrow{f\phi_1} fa$$

where $\beta(\phi_2) = \text{id}$. Since f is a fibration in CAT/$\mathbb{C}$ one can take a cartesian lift ϕ_3 of ϕ_2, and then the composite

$$a_2 \xrightarrow{\phi_3} a_1 \xrightarrow{\phi_1} a$$

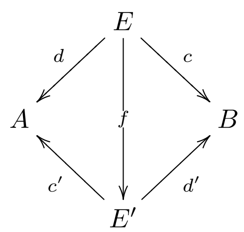
is a cartesian lift for ϕ.

2.3. 2-sided discrete fibrations and duality involutions. In this subsection $\mathcal{K}$ is a finitely complete 2-category. We shall now recall the notion of 2-sided discrete fibration of [**Str74b**] and an associated yoneda lemma, and then define a notion of duality involution for $\mathcal{K}$. Recall first the bicategory Span($\mathcal{K}$):

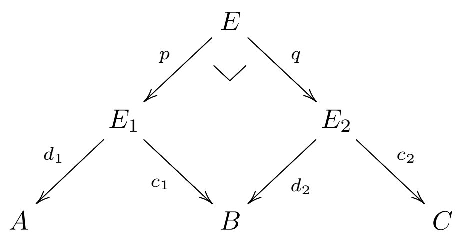
- objects: those of $\mathcal{K}$.
- a morphism $A \to B$ in Span($\mathcal{K}$) is a triple (d, E, c)

  $$A \xleftarrow{d} E \xrightarrow{c} B$$

  and is called a *span* from A to B. When the maps d and c are understood we may abuse notation a little and refer to the above span as E.
- a 2-cell between spans is a commutative diagram

  Composition of 2-cells is just composition in $\mathcal{K}$.
- composition of 1-cells is obtained by pulling back: the composite of (d_1, E_1, c_1) and (d_2, E_2, c_2)

  is (d_1p, E, c_2q), and this composite may be denoted as $E_2 \circ E_1$.

Notice that the homs Span($\mathcal{K}$)(A, B) being isomorphic to the underlying category of $\mathcal{K}/(A \times B)$ are in fact 2-categories, and that pullback-composition is 2-functorial. Given a map $f : A \to B$ in $\mathcal{K}$, we have spans $(1_A, A, f)$ and $(f, A, 1_A)$ which we denote by f and f^{rev} respectively. Note also that $f \dashv f^{\mathrm{rev}}$ in $\mathrm{Span}(\mathcal{K})$ and that for any span (d, E, c) one has $E \cong c^{\mathrm{rev}} \circ d$.

A span (d, E, c) from A to B in CAT is a *discrete fibration from A to B* when it satisfies:

(1) $\forall f : a \to d(e)$, $\exists! \overline{f}$ in E with codomain e such that $d(\overline{f}) = f$ and $c(\overline{f}) = 1_{c(e)}$.
(2) $\forall g : c(e) \to b$, $\exists! \overline{g}$ in E with domain e such that $d(\overline{f}) = 1_d(e)$ and $c(\overline{f}) = f$.
(3) $\forall h : e_1 \to e_2$ in E, the composite $\overline{d(h)} \circ \overline{c(h)}$ is defined and equal to h.

More generally, a span (d, E, c) from A to B in $\mathcal{K}$ is a *discrete fibration from A to B* when for all $X \in \mathcal{K}$

$$\mathcal{K}(X, A) \xleftarrow{\mathcal{K}(X,d)} \mathcal{K}(X, E) \xrightarrow{\mathcal{K}(X,c)} \mathcal{K}(X, B)$$

is a discrete fibration from $\mathcal{K}(X, A)$ to $\mathcal{K}(X, B)$. We define a category $\mathrm{DFib}(\mathcal{K})(A, B)$ whose objects are discrete fibrations from A to B in $\mathcal{K}$, and morphisms are morphisms of the underlying spans[6]. In particular a map $p : E \to B$ is a *discrete fibration* when the span

$$B \xleftarrow{p} E \longrightarrow 1$$

is a discrete fibration, and a *discrete opfibration* when the span

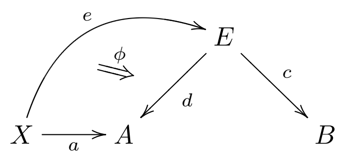
$$1 \longleftarrow E \xrightarrow{p} B$$

is a discrete fibration.

Theorem 2.11. *In a finitely complete 2-category $\mathcal{K}$:*

(1) *Given a discrete fibration E from A to B and a map $f : C \to A$, the span $E \circ f$ is a discrete fibration from C to A.*
(2) *Given a discrete fibration E from A to B and a map $g : D \to A$, the span $g^{\mathrm{rev}} \circ E$ is a discrete fibration from A to D.*
(3) *If (d, E, c) is a discrete fibration from A to B then d is a fibration and c is an opfibration.*
(4) *If $f : A \to C$ and $g : B \to C$, then the span from A to B obtained from the lax pullback f/g is a discrete fibration from A to B.*

Proof. Because of the representability of the notions involved, to prove (1), (2) and (4) it suffices to verify each statement in the case $\mathcal{K} = \mathrm{CAT}$, and in this case the direct verifications are straight forward. As for (3) it suffices to show that d is a fibration, because this result interpreted in $\mathcal{K}^{\mathrm{co}}$ says that c is an opfibration. Consider



and take $\varepsilon : e_1 \to e$ to be the unique 2-cell such that $d\varepsilon = \phi$ and $c\varepsilon = \mathrm{id}$. We will now show that ε is d-cartesian. Let $z : Z \to X$, $g : e_2 \to ez$ and $f : pe_2 \to az$ be such that

[6] A 2-cell between maps of discrete fibrations is necessarily an identity

$pg = (\phi z)f$. We must produce $\delta : e_2 \to e_1 z$ unique such that $d\delta = f$ and $(\varepsilon z)\delta = g$. Since (d, E, c) is a discrete fibration we can factor g uniquely as

$$e_2 \xrightarrow{g_1} e_3 \xrightarrow{g_2} ez$$

where

$$dg_1 = \text{id} \quad cg_1 = cg \quad dg_2 = dg \quad cg_2 = \text{id}$$

and we can take $h : e_4 \to e_1 z$ unique such that $dh = f$ and $ch = \text{id}$. Since $d((\varepsilon z)h) = (\phi z)f = dg$ and $c((\varepsilon z)h) = \text{id}$, we have $g_2 = (\varepsilon z)h$ and so $e_3 = e_4$. Thus we take δ to be the composite

$$e_2 \xrightarrow{g_1} e_3 \xrightarrow{h} e_1 z$$

because $d\delta = (dh)(dg_1) = f$ and $(\varepsilon z)\delta = (\varepsilon z)hg_1 = g_2 g_1 = g$. To see that δ is unique, let $\delta' : e_2 \to e_1 z$ be such that $d\delta' = f$ and $(\varepsilon z)\delta' = g$. Factor δ' uniquely as

$$e_2 \xrightarrow{\delta_1} e_5 \xrightarrow{\delta_2} e_1 z$$

where

$$d\delta_1 = \text{id} \quad c\delta_1 = c\delta' \quad d\delta_2 = f \quad c\delta_2 = \text{id}$$

Then δ_1 is forced to be g_1 since $d\delta_1 = \text{id} = dg_1$ and $c\delta_1 = c\delta' = c((\varepsilon z)\delta') = cg = cg_1$, and δ_2 is forced to be h since $d\delta_2 = f = dh$ and $c\delta_2 = \text{id} = ch$. □

In particular notice that by (3) discrete fibrations are fibrations and discrete opfibrations are opfibrations as one would hope.

We now recall an analogue of the yoneda lemma for 2-sided discrete fibrations. Let $f : A \to B$ be in $\mathcal{K}$ a finitely complete 2-category. Denote by

$$\begin{array}{ccc} f/B & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B \end{array} \qquad\qquad \begin{array}{ccc} B/f & \xrightarrow{q'} & A \\ {\scriptstyle p'}\downarrow & \overset{\lambda'}{\Rightarrow} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

the defining lax pullback squares for f/B and B/f, define $i_f : A \to f/B$ to be the unique map such that $pi_f = 1_A$, $qi_f = f$ and $\lambda i_f = \text{id}$, and $j_f : A \to B/f$ to be the unique map such that $p'j_f = f$, $q'j_f = 1_A$ and $\lambda' j_f = \text{id}$.

Theorem 2.12. *Let $\mathcal{K}$ be a finitely complete 2-category and $f : A \to B$ be in $\mathcal{K}$.*

(1) *(yoneda lemma): for any span (d_1, E_1, c_1) from X to A and discrete fibration (d_2, E_2, c_2) from X to B, a map of spans*

$$f/B \circ E_1 \to E_2$$

is determined uniquely by its composite with

$$i_f \circ \text{id} : f \circ E_1 \to f/B \circ E_1$$

(2) *(coyoneda lemma): for any span (d_1, E_1, c_1) from A to X and discrete fibration (d_2, E_2, c_2) from B to X, a map of spans*

$$E_1 \circ B/f \to E_2$$

is determined uniquely by its composite with

$$\text{id} \circ j_f : E_1 \circ f^{\text{rev}} \to E_1 \circ B/f$$

PROOF. The coyoneda lemma is the yoneda lemma in $\mathcal{K}^{co}$, so it suffices to prove the yoneda lemma. By the representability of the notions involved it suffices to prove this result for the case $\mathcal{K} = \mathrm{CAT}$. In this case the head of the span $f/B \circ E_1$ can be described as follows:

- objects are 4-tuples $(e, a, \beta : fa \to b, b)$ where $e \in E_1$ and $c_1 e = a$.
- an arrow
$$(\varepsilon, \alpha, \beta) : (e_1, a_1, \beta_1, b_1) \to (e_2, a_2, \beta_2, b_2)$$
consists of maps $\varepsilon : e_1 \to e_2$, $\alpha : a_1 \to a_2$ and $\beta : b_1 \to b_2$, such that $c_1\varepsilon = \alpha$ and $\beta\beta_1 = \beta_2 f(\alpha)$.

and the left and right legs of the span send $(\varepsilon, \alpha, \beta)$ described above to $d_1\varepsilon$ and β respectively. The image of $i_f \circ \mathrm{id}$ is the full subcategory given by the (e, a, β, b) such that $\beta = \mathrm{id}$. Let $\phi : f/B \circ E_1 \to E_2$. For any object (e, a, β, b) we have a map

$$(1, 1, \beta) : (e, a, \mathrm{id}, fa) \to (e, a, \beta, b)$$

and since $d_2\phi(1, 1, \beta) = \mathrm{id}$ and $c_2\phi(1, 1, \beta) = \beta$, $\phi(e, a, \beta, b)$ and $\phi(1, 1, \beta)$ are defined as the unique "right" lift of $(\phi(e, a, 1_{fa}, fa), \beta)$ since E_2 is a discrete fibration. For any arrow $(\varepsilon, \alpha, \beta)$ as above, we have a commutative square

$$\begin{array}{ccc} \phi(e_1, a_1, \mathrm{id}, fa_1) & \xrightarrow{\phi(1,1,\beta_1)} & \phi(e_1, a_1, \beta_1, b_1) \\ {\scriptstyle \phi(\varepsilon,\alpha,f\alpha)}\downarrow & & \downarrow{\scriptstyle \phi(\varepsilon,\alpha,\beta)} \\ \phi(e_2, a_2, \mathrm{id}, fa_2) & \xrightarrow[\phi(1,1,\beta_2)]{} & \phi(e_2, a_2, \beta_2, b_2) \end{array}$$

in E_2, but from the proof of theorem(2.11) we know that $\phi(1, 1, \beta_1)$ is d_2-opcartesian and so the rest of the above square is determined uniquely by $\phi(\varepsilon, \alpha, f\alpha)$. □

The yoneda lemma of [**Str74b**] is theorem(2.12)(1) in the case where E_1 is the identity span. We use an analogue of this more general result in the proof of theorem(8.6).

COROLLARY 2.13. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & C & \end{array}$$

be in $\mathcal{K}$. Then composition with i_f

$$\mathcal{K}(f/B, C)(gp, hq) \xrightarrow{(-)\circ i_f} \mathcal{K}(A, C)(g, hf)$$

is a bijection with inverse given by pasting with λ.

PROOF. 2-cells $gp \to hq$ are in bijection with span maps $f/B \to g/h$ by the definition of g/h. By theorem(2.12) these are in bijection with span maps $f \to g/h$ which by the definition of g/h are in bijection with 2-cells $g \to hf$. This composite bijection is clearly $(-)\circ i_f$. By the definition of i_f the inverse of this bijection is given by pasting with λ. □

In order to define duality involutions we shall now return to our discussion of $\mathrm{Span}(\mathcal{K})$. The operation

$$B \xleftarrow{c} E \xrightarrow{d} A \quad \mapsto \quad A \xleftarrow{d} E \xrightarrow{c} B$$

of reversing spans is part of an isomorphism of bicategories

$$(-)^{\mathrm{rev}} : \mathrm{Span}(\mathcal{K})^{\mathrm{op}} \to \mathrm{Span}(\mathcal{K})$$

which is the identity on objects and provides isomorphisms of 2-categories on the homs. Cartesian product on $\mathcal{K}$ extends to a tensor product on $\mathrm{Span}(\mathcal{K})$, and this is compatible with $(-)^{\mathrm{rev}}$ because the operation

$$A{\times}B \xleftarrow{(p,q)} E \xrightarrow{r} C \quad \mapsto \quad A \xleftarrow{p} E \xrightarrow{(q,r)} B{\times}C$$

provides the object part of isomorphisms of 2-categories

$$\mathrm{Span}(\mathcal{K})(A{\times}B, C) \cong \mathrm{Span}(\mathcal{K})(A, B^{\mathrm{rev}}{\times}C) \tag{1}$$

which are pseudo natural in A, B and C[7]. Notice that $(-)^{\mathrm{rev}}$ does not restrict to 2-sided discrete fibrations: for instance the map d in a discrete fibration (d, E, c) is a fibration although not in general an opfibration. However from theorem(2.11)(1)-(2) and using the lifting properties from the definition of 2-sided discrete fibration, one can verify that the formation of categories of 2-sided discrete fibrations provides a pseudo functor

$$\mathrm{DFib}(\mathcal{K})(-,-) : \mathcal{K}^{\mathrm{coop}} \times \mathcal{K}^{\mathrm{op}} \to \mathrm{CAT}$$

the pseudoness arising because of span composition.

Definition 2.14. Let $\mathcal{K}$ be a finitely complete 2-category. A *duality involution* for $\mathcal{K}$ consists of a 2-functor

$$(-)^{\circ} : \mathcal{K}^{\mathrm{co}} \to \mathcal{K}$$

such that $((-)^{\circ})^{\circ} = \mathrm{id}$ and for all A, B and C in $\mathcal{K}$ equivalences of categories

$$\mathrm{DFib}(\mathcal{K})(A \times B, C) \simeq \mathrm{DFib}(\mathcal{K})(A, B^{\circ} \times C)$$

pseudo natural in A, B and C.

Example 2.15. Let (d, E, c) be a discrete fibration from A to B in CAT. Then for each $a \in A$ and $b \in B$ one can define

$$E(a,b) = \{e \in E : \quad de = a \text{ and } ce = b\}$$

which is contravariant in a and covariant in b and so defines a functor $A^{\mathrm{op}} \times B \to \mathrm{SET}$. On the other hand given $P : A^{\mathrm{op}} \times B \to \mathrm{SET}$ one can define a discrete fibration from A to B whose domain is the category

- objects: triples (a, e, b) where $a \in A$, $b \in B$ and $e \in P(a, b)$.
- a morphism $(a_1, e_1, b_1) \to (a_2, e_2, b_2)$ consists of $\alpha : a_2 \to a_1$ and $\beta : b_1 \to b_2$ such that $P(\alpha, \beta)(e_1) = e_2$.

These constructions define the object maps of equivalences of categories

$$\mathrm{DFib}(\mathrm{CAT})(A, B) \simeq [A^{\mathrm{op}} \times B, \mathrm{SET}]$$

which are pseudo natural in A and B. Using these equivalences and the isomorphisms

$$[(A \times B)^{\mathrm{op}} \times C, \mathrm{SET}] \cong [A^{\mathrm{op}} \times (B^{\mathrm{op}} \times C), \mathrm{SET}]$$

[7] It's the sense in which these isomorphisms are natural in B which neccessitates the $(-)^{\mathrm{rev}}$ here. The objects B and B^{rev} are of course equal.

one can exhibit $(-)^{\mathrm{op}}$ as a duality involution for CAT.

Example 2.16. Let $\mathcal{E}$ be a category with finite limits. Define A° for $A \in \mathrm{Cat}(\mathcal{E})$ by interchanging the source and target maps. For $A, B \in \mathrm{Cat}(\mathcal{E})$ there is a forgetful functor

$$\mathrm{DFib}(\mathrm{Cat}(\mathcal{E}))(A, B) \to \mathrm{Span}(\mathcal{E})(A_0, B_0)$$

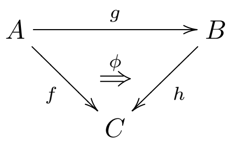
which is monadic. The relevant monad on $\mathrm{Span}(\mathcal{E})(A_0, B_0)$ is given by span composition $B \circ - \circ A$ regarding A and B as monads in $\mathrm{Span}(\mathcal{E})$. One can verify these facts directly in the case $\mathcal{E}$=SET. Using the internal language of $\mathcal{E}$ (again see [**Joh02**]) this verification may in fact be interpretted in $\mathcal{E}$ to give a proof of the general result. See also [**Str80a**] for a derivation of these facts from the viewpoint of more general 2-sided fibrations. From this monadicity and the description of the relevant monad, the isomorphisms (1) above (in the case $\mathcal{K}$=$\mathcal{E}$) lift through the forgetful functors to provide equivalences of categories

$$\mathrm{DFib}(\mathrm{Cat}(\mathcal{E}))(A \times B, C) \simeq \mathrm{DFib}(\mathrm{Cat}(\mathcal{E}))(A, B^{\circ} \times C)$$

pseudo natural in A, B and C.

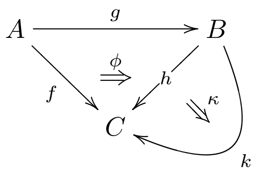
2.4. Left extensions and left liftings. As we shall see later in this paper, to express the cocompleteness of an object A of a finitely complete 2-category $\mathcal{K}$, one requires pointwise left extensions and left liftings. We recall these notions and some results about left extending along fully faithful maps and opfibrations in this subsection.

A 2-cell



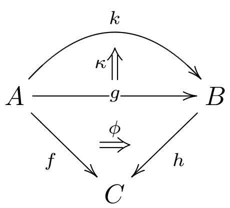
in a 2-category $\mathcal{K}$ exhibits h as a *left extension* of f along g when $\forall k$ pasting with ϕ:

$\kappa : h \Rightarrow k \quad \mapsto$



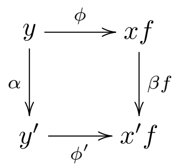
provides a bijection between 2-cells $h \Rightarrow k$ and 2-cells $f \Rightarrow kg$. This left extension is *preserved* by $j : C \to D$ when $j\phi$ exhibits jh as a left extension of jf along g, and is *absolute* when it is preserved by all arrows out of C. The 2-cell ϕ exhibits g as a *left lifting* of f along h when $\forall k$ pasting with ϕ:

$\kappa : g \Rightarrow k \quad \mapsto$

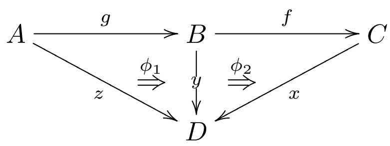


provides a bijection between 2-cells $g \Rightarrow k$ and 2-cells $f \Rightarrow hk$. This left lifting is *respected* by $j : D \to A$ when ϕj exhibits gj as a left lifting of fj along h and is absolute when it is respected by all arrows into C. Clearly a left extension in $\mathcal{K}$ is a left lifting in $\mathcal{K}^{\mathrm{op}}$. Applying the above definitions to the 2-category $\mathcal{K}^{\mathrm{co}}$ gives the notions of right extension and right lifting. Some basic elementary facts regarding left extensions in $\mathcal{K}$ are:

(1) If f is an isomorphism and $\phi : y \to xf$ then ϕ is a left extension along f iff ϕ is an isomorphism.
(2) If ϕ and ϕ' are left extensions along f, and β is the unique 2-cell making



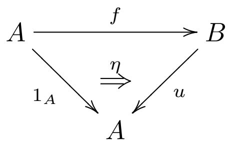
commute; then if α is an isomorphism so is β.
(3) If ϕ_1 is a left extension along g



then $\phi_2 : y \to xf$ is a left extension along f iff the composite is a left extension along fg.

and applying these observations to the various duals of $\mathcal{K}$ gives analogous statements for left liftings, right extensions and right liftings.

Example 2.17. The unit of an adjunction provides the most basic example of a left lifting and left extension, and this expresses its universal nature 2-categorically: let

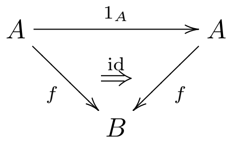


be a 2-cell in a 2-category $\mathcal{K}$, then it is an elementary exercise to show that the following are equivalent:

(1) η is the unit of an adjunction $f \dashv u$.
(2) η exhibits u as a left extension of 1_A along f and this left extension is preserved by f.
(3) η exhibits u as a left extension of 1_A along f and this left extension is absolute.

Applying this observation in $\mathcal{K}^{\mathrm{op}}$ one obtains two further equivalent conditions for η in terms of left liftings, and considering $\mathcal{K}^{\mathrm{co}}$ one obtains the corresponding conditions for the counit of an adjunction in terms of right extensions and right liftings.

Example 2.18. Another basic example of absolute left liftings is provided by fully faithful maps. A map $f : A \to B$ in $\mathcal{K}$ is *fully faithful* when for all $X \in \mathcal{K}$, $\mathcal{K}(X, f)$ is a fully faithful functor. It is almost a tautology that f is fully faithful iff the identity cell



exhibits 1_A as an absolute left lifting of f along itself.

These last two examples are very useful observations, and as an illustration of this we prove some results about adjunctions at this level of generality, which are very well known in the case $\mathcal{K} = \mathrm{CAT}$.

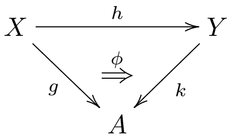
PROPOSITION 2.19. *In any 2-category $\mathcal{K}$:*

(1) *Left adjoints preserve left extensions.*
(2) *In an adjunction the left adjoint is fully faithful iff the unit is invertible.*

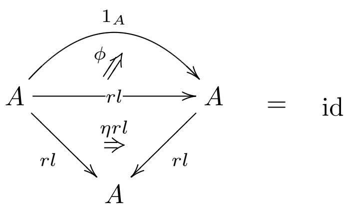
PROOF. (1): Suppose we have

$$A \underset{r}{\overset{l}{\rightleftarrows}} B \quad (l \dashv r)$$

in $\mathcal{K}$ with unit η, and suppose that



exhibits k as a left extension of g along h. Then composition with η gives a bijection between 2-cells $lk{\rightarrow}x$ and 2-cells $k{\rightarrow}rx$ since η is an absolute left lifting. Since ϕ is a left extension, pasting with it gives a bijection between 2-cells $k{\rightarrow}rx$ and 2-cells $g{\rightarrow}rxh$. Pasting η gives a bijection between these and 2-cells $lg{\rightarrow}xh$. The composite bijection is composition with $l\phi$, and so $l\phi$ is indeed a left extension.
(2): Suppose we have an adjunction as in (1) for which l is fully faithful. By the composability of left liftings, η exhibits 1_A as an absolute left lifting of 1_A along rl, and so ηrl exhibits rl as an absolute left lifting of rl along itself, and so there is a unique ϕ satisfying



Now

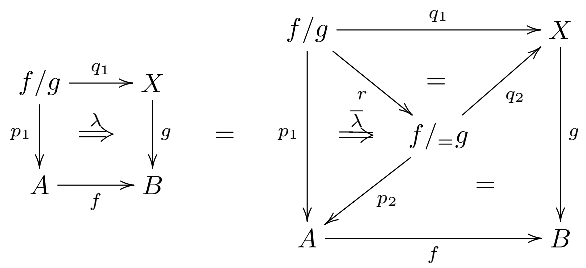
$$\eta\phi = (rl\phi)(\eta rl) = \mathrm{id}$$

and

$$(rl\phi)(rl\eta)\eta = (rl\phi)(\eta rl\eta = \eta$$

so that $\phi\eta = \mathrm{id}$ since η exhibits 1_A as a left lifting of 1_A along rl. The converse follows by a representable argument using the result in CAT. $\square$

EXAMPLE 2.20. Let $f : A \to B$ be an opfibration and $g : C \to B$ in a finitely complete 2-category $\mathcal{K}$. In the proof of theorem(2.7) carried out in $\mathcal{K}^{\mathrm{co}}$ so that it applies to opfibrations, we factored the lax pullback through the pullback

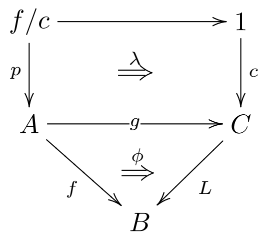


to obtain the f-opcartesian 2-cell $\overline{\lambda}$. Since the unique $\eta : 1 \to ir$ such that $p_1\eta = \overline{\lambda}$ and $q_1\eta = \mathrm{id}$ is the unit of an adjunction $r \dashv i$ by the proof of theorem(2.7), η exhibits i as an absolute left extension along r by the example(2.17). Thus $\overline{\lambda}$ exhibits p_2 as an absolute left extension of p_1 along r. This is the key observation for theorem(2.23) below.

As for the description of limits and colimits, the basic example is to take a functor $f : A \to B$ and then a left extension of f along $t_A : A \to 1$ is a colimit for f in B, and a right extension of f along t_A is a limit for f in B. In practise to compute a left extension L of f along $g : A \to C$ using colimits in B, one has the famous formula

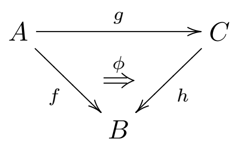
$$L(c) = \mathrm{colim}(\ f/c \longrightarrow A \xrightarrow{f} B\ )$$

due to Bill Lawvere. Diagramatically we have

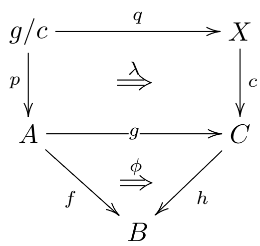


in CAT where λ is a lax pullback, ϕ exhibits L as a left extension, and Lawvere's formula means that the composite 2-cell exhibits Lc as a left extension along fp. This basic example suggests that the left extensions that arise in mathematical practise remain left extensions when pasted with lax pullbacks in this way.

To this end a 2-cell



in a finitely complete 2-category $\mathcal{K}$ is defined to be a *pointwise left extension* when for all $c : X \to C$, the composite



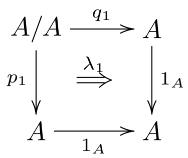
exhibits hc as a left extension of fp along q. Such a ϕ is automatically a left extension [**Str74b**]: to see this apply the definition with $c = 1_C$ and use lemma(2.21) below.

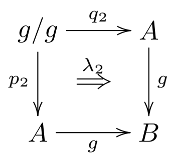
We shall now explain how pointwise left extending along fully faithful maps and opfibrations is well-behaved.

We explained what it means for $g : A \to B$ to be fully faithful in example(2.18). When $\mathcal{K}$ has finite limits one can form $H_g : A/A \to g/g$ as the unique map such that

$$p_2 H_g = p_1 \qquad q_2 H_g = q_1 \qquad \lambda_2 H_g = g\lambda_1$$

where



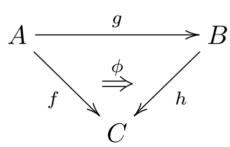


are lax pullbacks. Then g is fully faithful iff H_g is an isomorphism: this statemement is easily seen as true in the case $\mathcal{K} = \mathrm{CAT}$ by direct inspection, and the general case follows by a representable argument.

Lemma 2.21. *The bijection of corollary(2.13) sends 2-cells $gp \to hq$ which exhibit h as a left extension along q, to 2-cells which exhibit h as a left extension along f.*

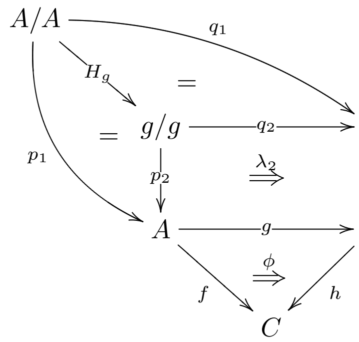
Proof. This bijection clearly respects composition with 2-cells $h \to h'$. □

Proposition 2.22. [**Str74b**] *If*



exhibits h as a pointwise left extension of f along g and g is fully faithful, then ϕ is invertible.

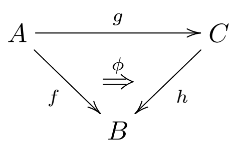
Proof. Since H_g is invertible the composite



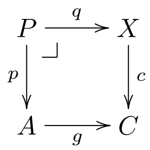
exhibits hg as a left extension of fp_1 along q_1, and so by lemma(2.21) ϕ exhibits f as a left extension along 1_A, whence ϕ is an isomorphism. □

When taking a pointwise left extension along an opfibration, lax pullbacks may be replaced by pullbacks for the sake of computation.

Theorem 2.23. [**Str74b**] *Let $\mathcal{K}$ be a finitely complete* 2*-category and*

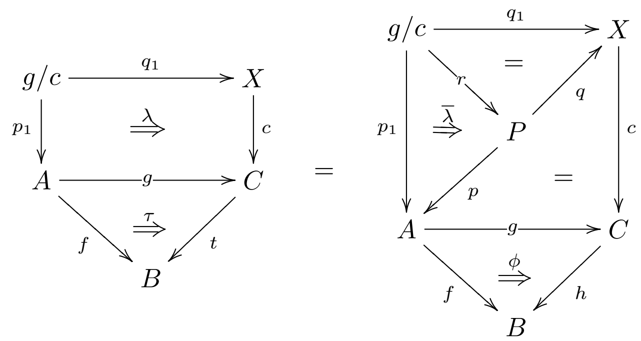


be in $\mathcal{K}$ such that g is an opfibration. Then ϕ exhibits h as a pointwise left extension of f along g iff for all $c : X \to C$ the 2*-cell ϕp where*



exhibits hc as a left extension of fp along q.

Proof. For $c : X \to C$ we have



by the factorisation of lax pullbacks described in example(2.20) where $\overline{\lambda}$ exhibits p as an absolute left extension of p_1 along r. Thus $f\overline{\lambda}$ exhibits fp as a left extension of fp_1 along r. Thus by the elementary properties of left extensions, the composite of

ϕ and λ exhibits hc as a left extension along q_1 iff ϕp exhibits hc as a left extension along q. □

The next result expresses the consequence that *pointwise* left extensions are "closed" under the operation of pasting with lax pullback squares, and pointwise left extensions along opfibrations are in addition closed under pasting with pullback squares.

Corollary 2.24. [**Str74b**] *Let $\mathcal{K}$ be a finitely complete 2-category and*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ & {}_{f}\searrow \overset{\phi}{\Rightarrow} \swarrow_{h} & \\ & B & \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle c} \\ A & \xrightarrow[g]{} & C \end{array}$$

be in $\mathcal{K}$. Suppose that ϕ exhibits h as a pointwise left extension of f along g.

(1) *If λ exhibits P as g/c then the composite of λ and ϕ exhibits hc as a pointwise left extension along q.*
(2) *If λ exhibits P as the pullback of g and c and g is an opfibration, then the composite of λ and ϕ exhibits hc as a pointwise left extension along q.*

Proof. In both (1) and (2) note that q is an opfibration. For (1): obtain q by first lax pulling back g along 1_C, which is the free opfibration on g, and then pulling back the result along c, which is an opfibration since opfibrations are pullback stable. For (2): q is an opfibration since opfibrations are pullback stable. Thus both results follow immediately from the previous theorem and the basic properties of pullbacks and lax pullbacks. □

3. Yoneda Structures

There are two main sources of examples of yoneda structures: (1) internal category theory; and (2) enriched category theory. Those that arise from internal category theory are more nicely behaved. They satisfy a further axiom, axiom(3*) of [**SW78**], and the left extensions that form part of the axiomatics are all pointwise in the sense discussed above. We restrict our attention to this special case in the following definition, and later isolate a further condition – cocompleteness of presheaf objects.

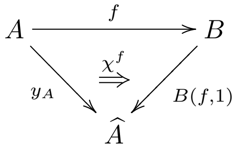
Definition 3.1. A *good yoneda structure* on a finitely complete 2-category $\mathcal{K}$ consists of

(1) A right ideal[8] of 1-cells called *admissible* maps. An object A of $\mathcal{K}$ is admissible when 1_A is so.[9]
(2) For each admissible object, an object $\widehat{A}$ and an admissible map
$$y_A : A \to \widehat{A}$$
is provided.

[8]A *right ideal* in a category is a class $\mathcal{I}$ of arrows such that $f \in \mathcal{I}$ implies $fg \in \mathcal{I}$ for any g for which the composite fg is defined.

[9]Notice that if B is admissible and $f : A \to B$, then f is admissible since the admissible maps form a right ideal and of course $f = 1_B f$.

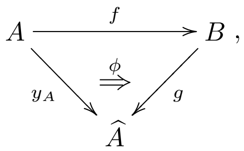
(3) For each $f : A \to B$ with both A and f admissible, an arrow $B(f,1)$ and a 2-cell



is provided.

This data must satisfy the following axioms:

(1) χ^f exhibits f as an absolute left lifting of y_A through $B(f,1)$.
(2) If A and $f : A \to B$ are admissible and



exhibits f as an absolute left lifting of y_A along g, then ϕ exhibits g as a pointwise left extension of y along f.

A yoneda structure in the sense of [**SW78**] is defined as above except: one replaces axiom(2) by an axiom which asserts only that χ^f is a left extension along f, and proposition(3.4) below is taken as an axiom. Moreover in [**SW78**] the hypothesis that $\mathcal{K}$ have finite limits is not needed.

Lemma 3.2. *Let A, f, g, ϕ be given as in axiom(2). If ϕ exhibits g as a left extension of y_A along f; then this left extension is pointwise and ϕ exhibits f as an absolute left lifting of y_A along g.*

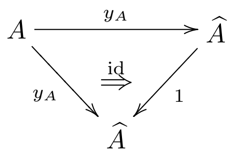
Proof. Suppose that ϕ is a left extension. Then since f is admissible there is an isomorphism $B(f,1) \cong g$ whose composite with χ^f is ϕ, whence ϕ is an absolute left lifting since χ^f is. Moreover ϕ is a pointwise left extension since χ^f is. □

Example 3.3. We shall see later in example(6.2), that CAT has the following good yoneda structure, which is *the* basic example. A functor $f : A \to B$ is admissible when $\forall a, b$, the homset $B(fa, b)$ is small. Admissible objects are locally small categories. For such a category A, $\widehat{A} = [A^{\mathrm{op}}, \mathrm{Set}]$ and y_A is the yoneda embedding. For a functor $f : A \to B$ such that A and f are admissible, $B(f,1)(b)(a) = B(fa, b)$ and $(\chi^f)_a : A(-,a) \to B(f-, fa)$ is given by the arrow maps of f.

Inspired by this example one should regard the 2-cell χ^f as "the arrow maps of f" and indeed the notation is selected to encourage this idea. Given $f : A \to B$ with A and f admissible and $x : X \to B$ we denote by $B(f,x)$ the composite $B(f,1)x$. When in addition B is admissible, denote by res_f the map $\widehat{A}(y_B f, 1)$. Given $z : Z \to A$ with Z admissible the following proposition justifies denoting the composite $\mathrm{res}_z B(f,1)$ as $B(fz,1)$. In fact the next two results enforce these hom-set interpretations.

Proposition 3.4. [**SW78**]

(1) *If A is admissible then*

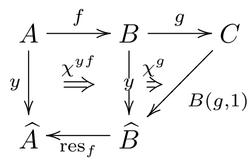


exhibits 1 *as a left extension of* y_A *along* y_A.

(2) *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

where A, B, f and g are admissible, then the composite



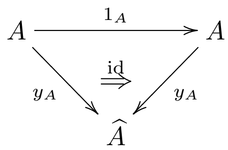
exhibits $res_f B(g,1)$ *as a left extension of* y_A *along* gf.

Proof. Trivially, the identity 2-cell in (1) exhibits y_A as an absolute left lifting of y_A along $1_{\widehat{A}}$. By the composability of left liftings, the composite 2-cell in (2) exhibits gf as an absolute left lifting of y_A. The result follows from axiom(2). □

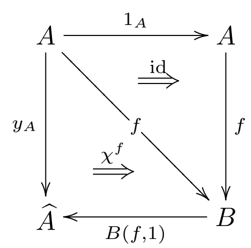
Corollary 3.5. [**SW78**]

(1) *If A is admissible then y_A is fully faithful.*
(2) *For $f : A \to B$ with A and f admissible, f is fully faithful iff χ^f is an isomorphism.*

Proof. (1): The identity cell



is trivially a left extension and thus an absolute left lifting by lemma(3.2). The result follows by example(2.18).
(2)(⇒): by proposition(2.22).
(2)(⇐): Since χ^f is an isomorphism and y_A is fully faithful, χ^f exhibits 1_A as an absolute left lifting of y_A along $B(f,f)$. Thus in the following diagram

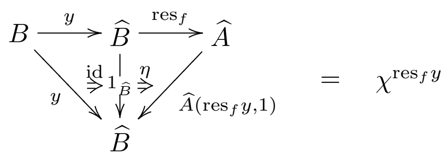


id exhibits 1_A as an absolute left lifting along itself, and so the result follows by example(2.18). □

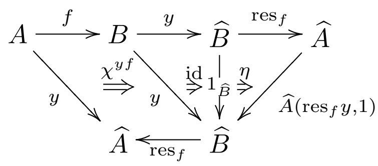
The notation res_f is also meant to be evocative: in the case of example(3.3) res_f corresponds to restriction along (that is, precomposition with) f^{op}. So one is immediately lead to ask when res_f has left and right adjoints. The case of the right adjoints is the easiest and we shall describe this now. The treatment of the left adjoints is provided in theorem(3.20) at the end of this section.

Definition 3.6. [**FS95**] An object C of $\mathcal{K}$ is *small* when both C and $\widehat{C}$ are admissible.

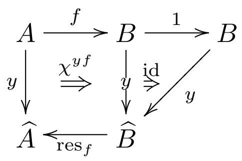
Let $f : A \to B$ with A small and B admissible. Let η be the unique 2-cell such that



Then η exhibits ran_f as a left extension along res_f. Now observe that the composite



exhibits $\mathrm{res}_f \widehat{A}(\mathrm{res}_f y, 1)$ as a left extension by proposition(3.4), and so the left extension η is preserved by res_f. Notice also that the composite



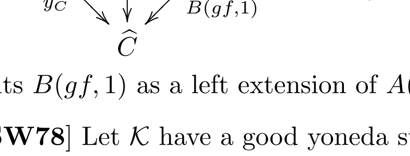
exhibits $\mathrm{res}_f y$ as a pointwise left extension whence $\mathrm{res}_f y \cong B(f, 1)$. We have proved:

Proposition 3.7. [**SW78**] *Given $f : A \to B$ with A small and B admissible, res_f has right adjoint $\widehat{A}(\mathrm{res}_f y, 1) \cong \widehat{A}(B(f, 1), 1)$.*

The remainder of this subsection is devoted to the discussion of internal colimits. First we shall give the abstract definitions of colimits and cocompleteness. In preparation for this consider

$$C \xrightarrow{f} A \xrightarrow{g} B$$

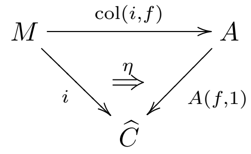
with C, f, g admissible, denote by χ_f^g the unique 2-cell such that



and note that χ_f^g exhibits $B(gf, 1)$ as a left extension of $A(f, 1)$ along g.

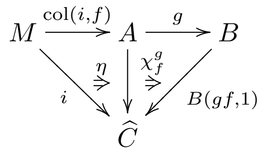
Definition 3.8. [**SW78**] Let $\mathcal{K}$ have a good yoneda structure.

(1) Consider $i : M \to \widehat{C}$ and $f : C \to A$ with M and f admissible, and C small. A *colimit* of f *weighted by* i consists of an admissible map $\mathrm{col}(i,f) : M \to A$ together with



which exhibits $\mathrm{col}(i,f)$ as an absolute left lifting of i through $A(f,1)$.

(2) $\mathrm{col}(i,f)$ is *preserved by* an admissible map $g : A \to B$ when the composite 2-cell



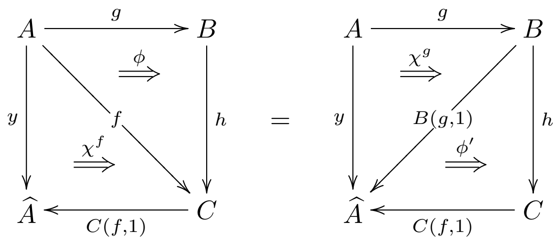
exhibits $g(\mathrm{col}(i,f))$ as the absolute left lifting of i along $B(gf,1)$.

(3) A is *small cocomplete* when $\mathrm{col}(i,f)$ exists for all $i : M \to \widehat{C}$ and $f : C \to A$ with M and C small and f admissible.

(4) $g : A \to B$ is *small cocontinuous* when for all $i : M \to \widehat{C}$ and $f : C \to A$ with M and C small and f admissible such that $\mathrm{col}(i,f)$ exists, $\mathrm{col}(i,f)$ is preserved by g.

At first glance the above definition may seem very abstract so we shall now reconcile it with the corresponding familiar notions. The key procedure that enables one to do this is a bijection between 2-cells ϕ and ϕ' so that

(2)



where A, f and g are assumed admissible. Equation(2) does indeed establish a bijection $\phi \mapsto \phi'$ because χ^g is a left extension and χ^f is a left lifting.

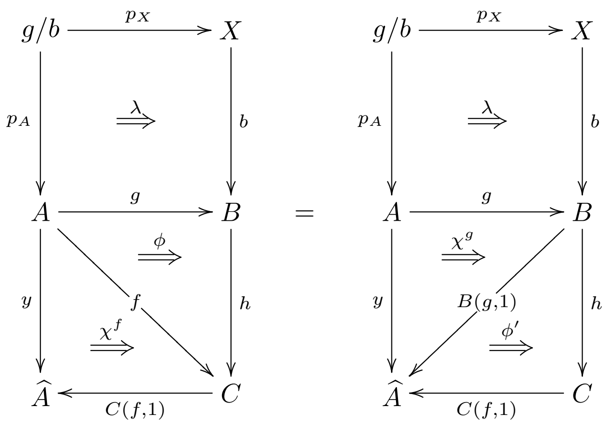
Lemma 3.9. (1) *ϕ exhibits h as a left extension of f along g iff ϕ' exhibits h as a left lifting of $B(g,1)$ along $C(f,1)$.*

(2) *ϕ exhibits h as a pointwise left extension of f along g iff ϕ' exhibits h as an absolute left lifting of $B(g,1)$ along $C(f,1)$.*

(3) *ϕ exhibits g as an absolute left lifting of f along h iff ϕ' is an isomorphism.*

Proof. (1): the result follows since the bijection $\phi \mapsto \phi'$ respects composition with 2-cells $h \to k$.

(2): For any $b : X \to B$ we have



Now ϕ' is an absolute left lifting along $C(f,1)$ iff for all b and $c : X \to C$, pasting with ϕ' gives a bijection $\mathcal{K}(X,C)(bh,c) \cong \mathcal{K}(X,\widehat{A})(B(g,b),C(f,c))$. However the composite of χ^g and λ is a left extension along p_X, so this is the same as saying that pasting with the composite 2-cell (on either side of the above equation) gives a bijection $\mathcal{K}(X,C)(bh,c) \cong \mathcal{K}(g/b,\widehat{A})(y_A p_A, C(f,cp_X))$ for all b and c. However χ^f is an absolute left lifting along $B(f,1)$, so this is the same as saying that pasting with the composite of ϕ and λ gives a bijection $\mathcal{K}(X,C)(bh,c) \cong \mathcal{K}(X,\widehat{A})(fp_A, cp_X)$ for all b and c, and this is by definition the statement that ϕ is a pointwise left extension.
(3): ϕ is an absolute left lifting iff composite 2-cell in equation(2) exhibits g as an absolute left lifting of f along $C(f,h)$, which by axiom(2) and lemma(3.2) is true iff this composite exhibits $C(f,h)$ as a left extension of y_A along g, which is true iff ϕ' is an isomorphism. □

Armed with this procedure we can now reconcile definition(3.8) with the usual notion of weighted colimit in terms of the hom-set notation.

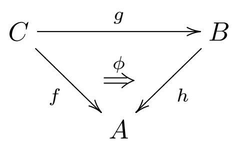
Corollary 3.10. *Consider $i : M \to \widehat{C}$ and $f : C \to A$ with M and f admissible, and C small. Then the colimit of f weighted by i exists iff there is an admissible map $\mathrm{col}(i,f) : M \to A$ together with an isomorphism*

$$A(\mathrm{col}(i,f),1) \cong \widehat{C}(i,C(f,1)).$$

Proof. By lemma(3.9)(3) the isomorphism is given by η' where η is the defining 2-cell of the weighted colimit. □

In the usual theory of weighted colimits one can express pointwise left extensions as weighted colimits in a canonical way. This fact is immediate in the present setting.

Corollary 3.11. *Suppose that C is small and B,$f : C \to A$, and $h : B \to A$ are admissible. Then there exists*

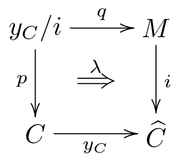


exhibiting h as a pointwise left extension of f along g iff the colimit of f weighted by $B(g,1)$ exists and there is an isomorphism

$$h \cong \mathrm{col}(B(g,1), f).$$

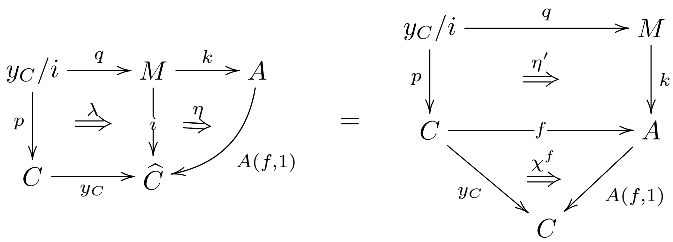
PROOF. By lemma(3.9)(2)-(3), the isomorphism is given by ϕ''. □

EXAMPLE 3.12. Another basic fact about colimits in ordinary category theory is that every presheaf is a colimit of representables. To appreciate the general analogue of this consider $i : M \to \widehat{C}$ where C is admissible. Then by proposition(3.4)(1) the lax pullback square



exhibits i as a pointwise left extension. The well known case of this is for the yoneda structure of example(3.3) with $M = 1$ and C small.

Another feature of the theory of colimits for CAT is that weighted colimits may be replaced by the more commonly used conical ones. This too has an analogue in a good yoneda structure in corollary(3.14) below. Consider $i : M \to \widehat{C}$ and $f : C \to A$ with M and f admissible, and C small. Since λ of the previous example is a left extension and χ^f a left lifting, the equations



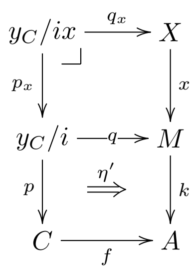
determine a bijection between 2-cells η and 2-cells η'.

LEMMA 3.13. (1) *η exhibits k as a left lifting of i along $A(f,1)$ iff η' exhibits k as a left extension of fp along q.*

(2) *η exhibits k as an absolute left lifting of i along $A(f,1)$ iff η' exhibits k as a pointwise left extension of fp along q.*

PROOF. (1): the bijection $\eta \mapsto \eta'$ clearly respects composition with 2-cells $k \to k'$.

(2): since q is an opfibration, η' is a pointwise left extension iff for every $x : X \to M$ the composite



exhibits kx as a left extension along q_x by theorem(2.23), which is equivalent by (1) to ηx exhibiting kx as a left lifting along $A(f, 1)$. □

An immediate consequence of lemma(3.13)(2) is:

COROLLARY 3.14. *Let $\mathcal{K}$ be a finitely complete 2-category equipped with a good yoneda structure and let*

$$i : M \to \widehat{C} \qquad f : C \to A$$

be in $\mathcal{K}$ where C is small and M and f are admissible. Let

$$p : y_C/i \to C \qquad q : y_C/i \to M$$

be the projections coming from the defining lax pullback square for y_C/i. Then the colimit of f weighted by i exists iff fp has a pointwise left extension along q as in

$$\begin{array}{ccc} y_C/i & \xrightarrow{q} & M \\ {\scriptstyle p}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle k} \\ C & \xrightarrow[f]{} & A \end{array}$$

such that k admissible. When this is the case $k \cong \mathrm{col}(i, f)$.

As a consequence of these results we can characterise small cocompleteness in terms of pointwise left extensions. For this we require a preliminary definition.

DEFINITION 3.15. A good yoneda structure is said to *exhibit small lax pullbacks* when for all lax pullbacks

$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

with A and C small and B admissible, P is small.

The paradigmatic example(3.3) and the yoneda structures discussed in example(6.5) all satisfy this condition. This can be easily seen using the characterisations of small and admissible objects obtained in section(6).

COROLLARY 3.16. *Let $\mathcal{K}$ have a good yoneda structure and $A \in \mathcal{K}$ be admissible.*

(1) *If A is small cocomplete then it admits all pointwise left extensions along any $f : X \to Z$ with X and Z small.*
(2) *The converse to (1) is true when $\mathcal{K}$'s yoneda structure exhibits small lax pullbacks.*

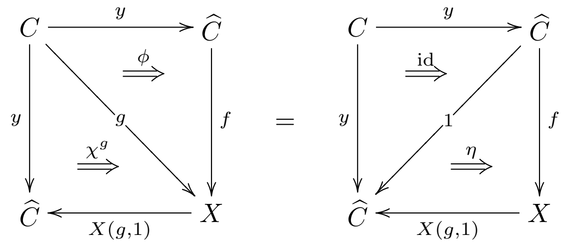
PROOF. (1): immediate from corollary(3.11).
(2): immediate from corollary(3.14), the admissibility of A and small lax pullbacks. □

However in order for the results on colimit completions to work, a slightly stronger notion of cocompleteness seems to be required.

Definition 3.17. An object A in a 2-category $\mathcal{K}$ with a good yoneda structure is *cocomplete* when it admits all pointwise left extensions along any $h : X \to Z$ such that X is small and Z is admissible. A morphism $f : A \to B$ is *cocontinuous* when it preserves all such pointwise left extensions.

Clearly for any yoneda structure exhibiting small lax pullbacks, cocomplete implies small cocomplete (and similarly for cocontinuity). For example(3.3) these notions are well known to coincide. However for the yoneda structures discussed in example(6.5), I don't know whether this is so.

Armed with these notions we now discuss colimit completions. From example(2.17) we know that left adjoints preserve left extensions, thus they preserve pointwise left extensions, and so by corollary(3.14) weighted colimits (as one would hope). On the other hand let C be small and suppose that $f : \widehat{C} \to X$ arises as a left extension of an admissible map g along y_C (via the 2-cell ϕ described below). Defining η as the unique 2-cell so that



then by lemma(3.9) η is an absolute left lifting since ϕ is a pointwise left extension. We have proved:

Lemma 3.18. [**SW78**] *Let C be small and suppose that $f : \widehat{C} \to X$ arises as the left extension of an admissible map g along y_C. Then $f \dashv X(g,1)$.*

Definition 3.19. A good yoneda structure is said to have *cocomplete presheaf objects* when for every small object A, the object $\widehat{A}$ is cocomplete.

Theorem 3.20. *Suppose $\mathcal{K}$ has a good yoneda structure which exhibits small lax pullbacks and has cocomplete presheaf objects.*

(1) *If C is small then $\widehat{C}$ is the colimit completion of C; in the sense that given X admissible and cocomplete, the adjunction*

$$\mathcal{K}(C,X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}(\widehat{C},X)$$

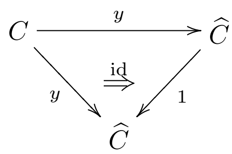
given by restriction and left extension along $y_C : C \to \widehat{C}$, restricts to an equivalence of categories

$$\mathcal{K}(C,X) \simeq \mathrm{CoCts}(\widehat{C},X)$$

where $\mathrm{CoCts}(\widehat{C},X)$ denotes the full subcategory of $\mathcal{K}(\widehat{C},X)$ consisting of the cocontinuous maps.

(2) *If $f : A \to B$ with A small and B admissible then res_f has left adjoint lan_f defined as the left extension of fy_B along y_A.*

Proof. (1): The asserted adjunction exists since $\mathcal{K}$'s yoneda structure has cocomplete presheaf objects. If $f : \widehat{C} \to X$ is cocontinuous then it preserves the left extension



so that it is the left extension along y_C of fy_C. Conversely, if f arises as a left extension along y_C, then it has a right adjoint by lemma(3.18), and so is cocontinuous.
(2): Defining $\mathrm{lan}_f : A \to B$ in this way note that since y_A is fully faithful we have $\mathrm{lan}_f y_A \cong y_B f$ and so by lemma(3.18) $\mathrm{lan}_f \dashv \mathrm{res}_f$. □

4. 2-toposes: Definition and Examples

Recall that a category $\mathcal{E}$ is an *elementary topos* when

(1) it has finite limits,
(2) is cartesian closed, and
(3) has a subobject classifier.

Finitely complete 2-categories have been discussed at length in section(2), and the definition of cartesian closed 2-category is the obvious direct analogue of its one dimensional counterpart: for each $A \in \mathcal{K}$, $(- \times A) : \mathcal{K} \to \mathcal{K}$ has a right 2-adjoint, which we denote as $[A, -]$.

The 2-categorical generalisation of subobject classifier is based on an important idea of Bill Lawvere regarding CAT: that *the category of sets is a generalised object of truth values.* This idea was expressed 2-categorically in the work of Ross Street [**Str74a**] [**Str80b**] as part of the notion of a fibrational cosmos. Following Street's approach, but allowing ourselves the luxury of a cartesian closed 2-category with a duality involution, one obtains the notion of 2-topos discussed here.

First we recall ordinary subobject classifiers. Let $\mathcal{E}$ be a locally small finitely complete category, and consider the functor

$$\mathrm{Sub}_0 : \mathcal{E}^{\mathrm{op}} \to \mathrm{SET}$$

which sends $E \in \mathcal{E}$ to the set of subobjects of E, and is given on arrows by pulling back. A *subobject classifier* is a representing object for Sub_0. This amounts to a monomorphism $\tau_0 : \Omega_{\bullet 0} \rightarrowtail \Omega_0$ in $\mathcal{E}$ such that for all $E \in \mathcal{E}$ the function

$$\mathcal{E}(E, \Omega_0) \to \mathrm{Sub}_0(E)$$

given by pulling back τ_0, is a bijection. Since $\mathcal{E}$ is locally small Sub_0 actually lands in Set. Moreover it is easily verified (or see [**MM91**] page 33 for example) that $\Omega_{\bullet 0}$ is a terminal object, so we will denote it as 1. Subobjects form not just a set but a poset, and as we shall recall below in example(4.3), τ_0 is the object part of an internal functor $\tau : 1 \to \Omega$.

The category Set as an object of CAT is analogous to Ω. To any functor $f : A \to \mathrm{Set}$ a version of the Grothendieck construction associates to f a discrete opfibration $G(f) : e(f) \to A$. Explicitly the category $e(f)$ can be described as having objects pairs (x, a) where $a \in A$ and $x \in f(a)$, and having arrows $(x_1, a_1) \to (x_2, a_2)$ which are maps $\alpha : a_1 \to a_2$ in A such that $f(\alpha)(x_1) = x_2$; and $G(f)(x, a) = a$. Any discrete opfibration into A with small fibres arises in this way, and so we have a fully faithful functor

$$G : \mathrm{CAT}(A, \mathrm{Set}) \to \mathrm{DFib}(1, A)$$

whose image consists of those discrete opfibrations with small fibres. To complete the analogy with subobject classifiers notice that G can be described more efficiently

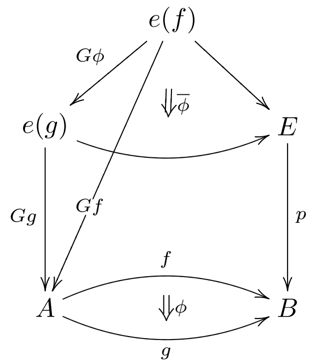
$$\begin{array}{ccc} e(f) & \longrightarrow & \mathrm{Set}_{\bullet} \\ {\scriptstyle G(f)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ A & \xrightarrow[f]{} & \mathrm{Set} \end{array}$$

as the process of pulling back the forgetful functor $U : \mathrm{Set}_{\bullet} \to \mathrm{Set}$ from the category $\mathrm{Set}_{\bullet}$ of pointed sets and functions which preserve the base point. The forgetful functor U is a discrete opfibration with small fibres, and we have just recalled the sense in which it is the universal such.

Let $p : E \to B$ be a discrete opfibration in a finitely complete 2-category $\mathcal{K}$. Then just as in the case of U above one has functors

$$G_{p,A} : \mathcal{K}(A, B) \to \mathrm{DFib}(1, A)$$

given by pulling back p. The effect of $G_{p,A}$ on morphisms is depicted as follows: given $\phi : f \Rightarrow g$ we have



where $\overline{\phi}$ is the unique lifting of ϕ, that is, $G(g)G(\phi) = G(f)$ and $\phi G(g) = p\overline{\phi}$.

Definition 4.1. Let $\mathcal{K}$ be a finitely complete 2-category. A discrete opfibration $\tau : \Omega_{\bullet} \to \Omega$ in $\mathcal{K}$ is *classifying* when the functors

$$G_{\tau,A} : \mathcal{K}(A, \Omega) \to \mathrm{DFib}(1, A)$$

are fully faithful for all $A \in \mathcal{K}$.

Example 4.2. By the discussion preceding definition(4.1) $U : \mathrm{Set}_{\bullet} \to \mathrm{Set}$ is a classifying discrete opfibration for CAT. There are many others. For example let λ be a regular cardinal, and in the above discussion replace Set by Set_{λ}, the category of sets of cardinality less than λ. Then the analogous forgetful functor $U_{\lambda} : \mathrm{Set}_{\bullet\lambda} \to \mathrm{Set}_{\lambda}$ is also a classifying discrete opfibration. In this case the image of

$$G_{U_{\lambda},A} : \mathrm{CAT}(A, \mathrm{Set}_{\lambda}) \to \mathrm{DFib}(1, A)$$

consists of the discrete opfibrations with fibres of cardinality less than λ. These examples illustrate that each classifying discrete opfibration provides a notion of smallness, and when one encodes category theory internally with the aid of a given classifying discrete opfibration as we do below, one has automatically accounted for size issues. Notice also that the case $\lambda = 2$ gives the ordinary subobject classifier for SET.

EXAMPLE 4.3. Let $\mathcal{E}$ be an elementary topos with subobject classifier denoted as $\tau_0 : 1 \to \Omega_0$. We shall now see that τ_0 is the object map of a classifying discrete opfibration $\tau : 1 \to \Omega$. For any $E \in \mathcal{E}$, $\mathrm{Sub}_0(E)$ has a maximum 1_E, and binary meets given by pullback

$$1 \xrightarrow{1_E} \mathrm{Sub}_0(E) \xleftarrow{\wedge} \mathrm{Sub}_0(E) \times \mathrm{Sub}_0(E)$$

and this structure is natural in E, and so by the yoneda lemma induces

$$1 \xrightarrow{\tau_0} \Omega_0 \xleftarrow{\wedge} \Omega_0 \times \Omega_0$$

in $\mathcal{E}$, satisfying the diagrams that express internally that τ_0 and $\wedge$ are the unit and multiplication for a commutative monoid structure on Ω_0 for which every element is idempotent. The poset structure is a consequence. For subobjects S_1 and S_2 of E, one has $S_1 \leq S_2$ iff $S_1 \wedge S_2 = S_1$, and so the corresponding poset structure on Ω induced by this is expressed in $\mathcal{E}$ by an equaliser

$$\Omega_1 \xrightarrow{\leq} \Omega_0 \times \Omega_0 \underset{\wedge}{\overset{\pi_1}{\rightrightarrows}} \Omega_0 \ .$$

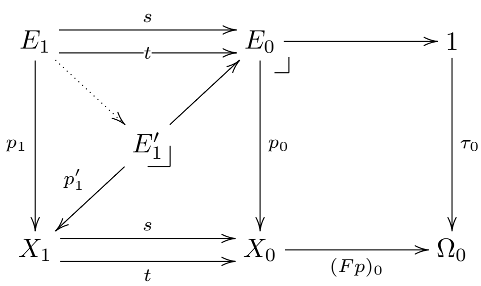
That is Ω_1 is the object of arrows of the internal category (in fact poset) Ω. From this explicit description it is immediate that:

(1) an internal functor $f : X \to \Omega$ amounts to a map $f_0 : X_0 \to \Omega_0$ such that $f_0 s \leq f_0 t$, where s and t are the source and target maps for the internal category X; and
(2) an internal natural transformation $\phi : f \Rightarrow g : X \to \Omega$ is unique it exists, and does so iff $f_0 \leq g_0$.

In particular $\tau : 1 \to \Omega$ is the internal functor corresponding to τ_0 by (1). For each X the functor

$$1 = \mathcal{E}(X, 1) \xrightarrow{\mathcal{E}(X,\tau)} \mathcal{E}(X, \Omega) \cong \mathrm{Sub}(X)$$

picks out the maximum element of $\mathrm{Sub}(X)$, and so is clearly a discrete opfibration. Thus τ is a discrete opfibration. We call a discrete opfibration $p : E \to X$ in $\mathrm{Cat}(\mathcal{E})$ a *cosieve* when p_0 is a monomorphism; the full subcategory of $\mathrm{DFib}(1, X)$ given by the cosieves into X is denoted by $\mathrm{Cosieve}(X)$. For example $\tau : 1 \to \Omega$ is a cosieve. Since cosieves are pullback stable $G_{\tau,X}$ factors through $\mathrm{Cosieve}(X)$ and we shall now see that it provides an equivalence $\mathrm{Cat}(\mathcal{E})(X, \Omega) \simeq \mathrm{Cosieve}(X)$. Given a cosieve $p : E \to X$ define $(Fp)_0 : X_0 \to \Omega_0$ to be the unique map corresponding to the subobject p_0



Since p is a discrete opfibration, the square involving p_1, p_0 and the source maps (both denoted as s) is a pullback, and so the monomorphism p_1 is classified by $(Fp)_0 s$. Denoting by E'_1 the pullback of p_0 along t, the resulting monomorphism

p'_1 is classified by $(Fp)_0 t$, p_1 factors through p'_1 (as indicated by the dotted arrow) and so $(Fp)_0 s \leq (Fp)_0 t$. Thus $(Fp)_0$ is the object map of a unique internal functor Fp. If the cosieve p factors through another cosieve q, then by construction we have $Fp \leq Fq$. By definition, $FG_{\tau,X}S = S$ for any subobject S and $G_{\tau,X}Fp \cong p$ for any cosieve p.

PROPOSITION 4.4. *Let $\mathcal{K}$ be a finitely complete 2-category, $\tau : \Omega_\bullet \to \Omega$ be a classifying discrete opfibration in $\mathcal{K}$, and $f : A \to \Omega$. Then τ_f defined by*

$$\begin{array}{ccc} A_\bullet & \longrightarrow & \Omega_\bullet \\ {\scriptstyle \tau_f}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ A & \xrightarrow[f]{} & \Omega \end{array}$$

is a classifying discrete opfibration iff f is fully faithful.

PROOF. By definition there is a natural isomorphism

$$\begin{array}{ccc} \mathcal{K}(X,A) & \xrightarrow{\mathcal{K}(X,f)} & \mathcal{K}(X,\Omega) \\ & {\scriptstyle G_{\tau_f,X}} \searrow \ \cong \ \swarrow {\scriptstyle G_{\tau,X}} & \\ & \mathrm{DFib}(1,X) & \end{array}$$

for all X, and $G_{\tau,X}$ is fully faithful. Thus $G_{\tau_f,X}$ is fully faithful iff $\mathcal{K}(X,f)$ is. □

EXAMPLE 4.5. Let $\mathcal{E}$ be an elementary topos with subobject classifier denoted as $\tau_0 : 1 \to \Omega_0$. A Lawvere-Tierney topology on $\mathcal{E}$ amounts to an idempotent monad j on Ω in $\mathrm{Cat}(\mathcal{E})$, such that j preserves $\wedge$. As a monad in a finitely complete 2-category we can take its Eilenberg-Moore object [**Str76**], part of which is the forgetful arrow $u^j : \Omega^j \to \Omega$. Since j is idempotent, u^j is fully faithful and by proposition(4.4)

$$\begin{array}{ccc} \Omega^j_\bullet & \longrightarrow & 1 \\ {\scriptstyle \tau_j}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ \Omega^j & \xrightarrow[u^j]{} & \Omega \end{array}$$

τ_j is a classifying discrete opfibration, and it is easily verified that $\Omega^j_\bullet$ is the terminal object (by the same argument that establishes $\Omega_\bullet = 1$ for subobject classifiers). The image of $G_{\tau_j,X}$ consists of those cosieves $p : E \to X$ such that p_0 is a monomorphism which is j-dense in the sense of topos theory. In fact $\tau_{j0} : 1 \to \Omega^j_0$ is the subobject classifier for the topos $\mathrm{Sh}_j(\mathcal{E})$ of sheaves for the topology j, and thus by example(4.3) τ_j is also a classifying discrete opfibration for $\mathrm{Cat}(\mathrm{Sh}_j(\mathcal{E}))$. This example could also have been obtained from example(4.3) via sheafification and the following theorem.

THEOREM 4.6. *Let*

$$\mathcal{A} \underset{S}{\overset{E}{\rightleftarrows}} \mathcal{B} \quad (\perp)$$

be a 2-adjunction, $\mathcal{A}$ and $\mathcal{B}$ be finitely complete 2-categories, and $\tau : \Omega_\bullet \to \Omega$ be a classifying discrete opfibration in $\mathcal{A}$. We write η and ε for the unit and counit of $E \dashv S$. If

(1) *E preserves pullbacks, and*
(2) *the naturality square*

$$\begin{array}{ccc} ES\Omega_\bullet & \xrightarrow{\varepsilon_{\Omega_\bullet}} & \Omega_\bullet \\ {\scriptstyle ES\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ ES\Omega & \xrightarrow[\varepsilon_\Omega]{} & \Omega \end{array}$$

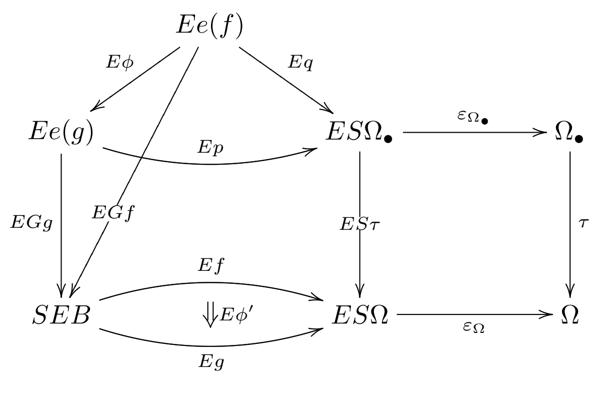
is a pullback

then $S\tau$ is a classifying discrete opfibration.

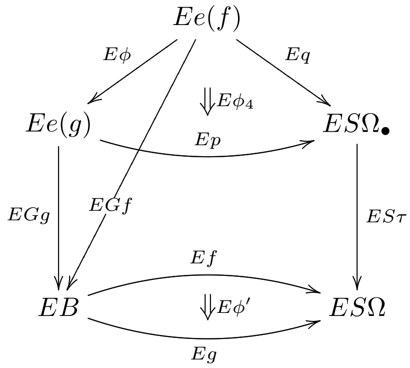
PROOF. Since discrete opfibrations can be defined representably, they are preserved by right 2-adjoints, and so it suffices to show that $S\tau$ is classifying. We abuse notation and denote $G_{S\tau,B}$ simply by G, and for $f : B \to S\Omega$ depict Gf by

$$\begin{array}{ccc} e(f) & \longrightarrow & S\Omega_\bullet \\ {\scriptstyle G(f)}\downarrow & \lrcorner & \downarrow{\scriptstyle S\tau} \\ B & \xrightarrow[f]{} & S\Omega \end{array}$$

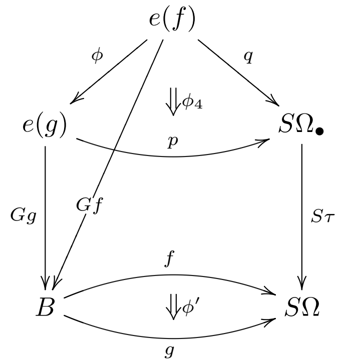
as in the discussion preceding definition(4.1). Suppose that $f, g : B \to S\Omega$ and $\phi : e(f) \to e(g)$ such that $G(g)\phi = G(f)$. We must show that there is a unique $\phi' : f \to g$ such that $G(\phi') = \phi$.



Since E preserves pullbacks, the naturality square in the above diagram is a pullback and τ is a classifying discrete opfibration, $E\phi$ induces a unique $\phi_2 : \varepsilon_\Omega E(f) \to \varepsilon_\Omega E(g)$ such that $G_{\tau,EB}(\phi_2) = E\phi$. By adjointness there is a unique ϕ' such that $\varepsilon_\Omega E(\phi') = \phi_2$. Now we must see that $G(\phi') = \phi$. Let $\phi_3 : \varepsilon_{\Omega_\bullet} E(q) \to \varepsilon_{\Omega_\bullet} E(p\phi)$ be the unique lifting of ϕ_2 through τ. By adjointness there is a unique $\phi_4 : q \to p\phi$ such that $E\phi_4 = \phi_3$, and by the uniqueness clause of the 2-dimensional universal property for the naturality pullback square, the diagram

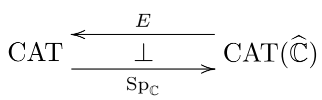


commutes, and so by adjointness



commutes also. To be more precise, paste the naturality square for ε onto the previous commuting diagram, apply S to the entire resulting diagram, and then precompose this with $\eta_{e(f)}$; the result of all this, because of the 2-naturality of η and adjunction equations, is the commutative diagram just obtained. That is, $G(f)\phi' = S(\tau)\phi_4$, and so ϕ_4 is the unique lifting of ϕ' through $S\tau$, so that $G\phi' = \phi$. To see that ϕ' is unique suppose that $\psi : f \to g$ such that $G\psi = \phi$. Then both $\varepsilon_\Omega E(\phi')$ and $\varepsilon_\Omega E(\psi)$ classify $E\phi$, and so they are equal. By adjointness one obtains $\phi' = \psi$ as required. ☐

Example 4.7. Let $\mathbb{C}$ be a small category and consider the 2-adjunction



given by

$$E(X) = \mathrm{el}(X^{\mathrm{op}})^{\mathrm{op}} \qquad \mathrm{Sp}_{\mathbb{C}}(Z) = [(\mathbb{C}/C)^{\mathrm{op}}, Z].$$

We shall now see that $\mathrm{Sp}_{\mathbb{C}}$ preserves classifying discrete opfibrations. By theorem(4.6) it suffices to show that if $p : A \to B$ is a discrete opfibration, then the naturality square

$$\begin{array}{ccc} E\mathrm{Sp}_{\mathbb{C}}A & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle E\mathrm{Sp}_{\mathbb{C}}p}\downarrow & & \downarrow{\scriptstyle p} \\ E\mathrm{Sp}_{\mathbb{C}}B & \xrightarrow[\varepsilon_B]{} & B \end{array}$$

is a pullback. To carry out this verification we need an explicit description of ε. For $A \in \mathrm{CAT}$ the category $E\mathrm{Sp}_{\mathbb{C}}A$ has:

- objects: pairs (C, a) where $C \in \mathbb{C}$ and $a : (\mathbb{C}/C)^{\mathrm{op}} \to A$.
- arrows: from $(C_1, a_1) \to (C_2, a_2)$ are pairs $(f, \overline{f})$, where $f : C_2 \to C_1$ and

$$\begin{array}{ccc} (\mathbb{C}/C_1)^{\mathrm{op}} & \xrightarrow{f^{!\mathrm{op}}} & (\mathbb{C}/C_2)^{\mathrm{op}} \\ & \overset{\overline{f}}{\Longrightarrow} & \\ {\scriptstyle a_1}\searrow & & \swarrow{\scriptstyle a_2} \\ & A & \end{array}$$

Then $\varepsilon_A(C, a) = a(1_C)$ and $\varepsilon_A(f, \overline{f})$ is $\overline{f}_{1_C}$. To see that the above naturality square is a pullback on objects, let $a \in A$ and $b : (\mathbb{C}/C)^{\mathrm{op}} \to B$ such that $b(1_C) = pa$. We must show that there is a unique $\overline{a} : (\mathbb{C}/C)^{\mathrm{op}} \to A$ such that $\overline{a}(1_C) = a$ and $f\overline{a} = b$. For $\gamma : C' \to C$ we have $b(t_\gamma) : pa \to b(\gamma)$ and so a unique $\overline{a}(t_\gamma) : a \to \overline{a}(\gamma)$ such that $p\overline{a}(t_\gamma) = b(t_\gamma)$. Functoriality and uniqueness for $\overline{a}$ follows from the uniqueness of the liftings involved in its description. To see that the above naturality square is a pullback on arrows, let $\alpha : a_1 \to a_2$ in A and $(\beta, \overline{\beta}) : (C_1, b_1) \to (C_2, b_2)$ in $E\mathrm{Sp}_{\mathbb{C}}B$ be such that $\overline{\beta}_{1_C} = p\alpha$. We must show that there is a unique $\overline{\alpha} : \overline{a_1} \to \overline{a_2}\beta^!$ such that $p\overline{\alpha} = \overline{\beta}$. For $\gamma : C \to C_1$ the square on the left

$$\begin{array}{ccc} pa_1 & \xrightarrow{b_1(t_\gamma)} & b_1\gamma \\ {\scriptstyle p\alpha}\downarrow & & \downarrow{\scriptstyle \overline{\beta}_\gamma} \\ pa_2 & \xrightarrow[b_2(t_\gamma)]{} & b_2\beta\gamma \end{array} \qquad \begin{array}{ccc} a_1 & \xrightarrow{\overline{a_1}(t_\gamma)} & \overline{a_1}(\gamma) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \overline{\alpha}_\gamma} \\ a_2 & \xrightarrow[\overline{a_2}(t_\gamma)]{} & \overline{a_2}(\beta\gamma) \end{array}$$

is commutative in B since it is the naturality square for $\overline{\beta}$ at t_γ, and $\overline{\alpha}_\gamma$ is defined as the unique map in A such that the square on the right is commutative in A and $p\overline{\alpha}_\gamma = \overline{\beta}_\gamma$. Naturality and uniqueness for $\overline{\alpha}$ follows from the uniqueness of the liftings involved in its description.

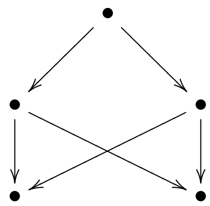
Example 4.8. Consider the case $\mathbb{C} = \mathbb{G}$, where $\mathbb{G}$ is the category presented as follows: the objects of $\mathbb{G}$ are natural numbers, for each $n \in \mathbb{N}$ there are generating maps

$$n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} n+1 \ ,$$

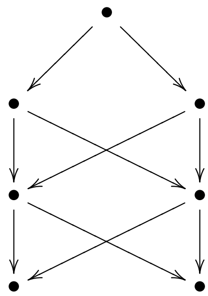
and these maps satisfy $\sigma\tau = \tau\tau$ and $\tau\sigma = \sigma\sigma$. The category $\widehat{\mathbb{G}}$ is the category of *globular sets* and $\mathrm{CAT}(\widehat{\mathbb{G}})$ is the 2-category of *globular categories* in the sense of [**Bat98b**]. A functor $(\mathbb{G}/n)^{\mathrm{op}} \to A$ is called an n-span in A, because in the case $n = 1$ such a functor is a span in A: the category $(\mathbb{G}/1)^{\mathrm{op}}$ is the poset which as a category looks like

$$\bullet \leftarrow \bullet \rightarrow \bullet$$

In fact all the $\mathbb{G}/n$ are posets. For example, the posets $(\mathbb{G}/2)^{\mathrm{op}}$ and $(\mathbb{G}/3)^{\mathrm{op}}$ look like



and



respectively. The globular category $\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$ plays a central role in [**Bat98b**], it is the globular category of higher spans, and in [**Str00**] it is seen as the globular analogue of the category of sets. In our language this last idea can be expressed as follows: by the previous example $\mathrm{Sp}_{\mathbb{C}}(U : \mathrm{Set}_{\bullet} \to \mathrm{Set})$ is a classifying discrete opfibration. Of course one may replace U by any classifying discrete opfibration in CAT (see example(4.2)) to get one in $\mathrm{CAT}(\widehat{\mathbb{G}})$ in this way.

Example 4.9. The examples described here are likely to be important in the study of the stabilisation hypothesis [**BD95**] in the globular setting. Consider the functor $(-+1) : \mathbb{G} \to \mathbb{G}$ which acts as $n \mapsto n+1$ on objects, and whose arrow map described by

$$n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} n+1 \quad \mapsto \quad n+1 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} n+2$$

Restriction and right extension along $(-+1)$ provides an adjunction between endofunctors of $\widehat{\mathbb{G}}$, and since these endofunctors are both right adjoints, they preserve pullbacks and so may be regarded as acting on category objects to provide a 2-adjunction

$$\mathrm{CAT}(\widehat{\mathbb{G}}) \underset{\Sigma}{\overset{D}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{CAT}(\widehat{\mathbb{G}}) \,.$$

Explicitly given a globular category X:

$$X_0 \leftleftarrows X_1 \leftleftarrows X_2 \leftleftarrows X_3 \leftleftarrows \cdots$$

DX is obtained by forgetting about X_0, and ΣX is

$$1 \leftleftarrows X_0 \leftleftarrows X_1 \leftleftarrows X_2 \leftleftarrows \cdots$$

The counit of this adjunction is an isomorphism, and so Σ preserves classifying discrete opfibrations by theorem(4.6). Thus for any classifying discrete opfibration $\tau : \Omega_{\bullet} \to \Omega$ in CAT, one obtains a sequence

$$\mathrm{Sp}_{\mathbb{G}}(\tau), \Sigma\mathrm{Sp}_{\mathbb{G}}(\tau), \Sigma^2\mathrm{Sp}_{\mathbb{G}}(\tau), \Sigma^3\mathrm{Sp}_{\mathbb{G}}(\tau), \ldots$$

of classifying discrete opfibrations in $\mathrm{CAT}(\widehat{\mathbb{G}})$.

Definition 4.10. A *2-topos* $(\mathcal{K}, (-)^{\circ}, \tau)$ is a finitely complete cartesian closed 2-category $\mathcal{K}$ equipped with a duality involution $(-)^{\circ}$ and a classifying discrete opfibration $\tau : \Omega_{\bullet} \to \Omega$.

We obtain our examples of 2-toposes from the above examples of classifying discrete opfibrations, since in each case the underlying $\mathcal{K}$ is of the form $\mathrm{Cat}(\mathcal{E})$ for $\mathcal{E}$ a finitely complete category, and so comes with a canonical duality involution by example(2.16).

5. Yoneda structures from 2-toposes

Let $(\mathcal{K}, (-)^\circ, \tau)$ be a 2-topos. The purpose of this section is to exhibit a good yoneda structure on $\mathcal{K}$. This construction is due to Ross Street [**Str74a**] [**Str80a**].

For A in $\mathcal{K}$ we define

$$\widehat{A} = [A^\circ, \Omega]$$

and so we have a 2-functor $\widehat{(-)} : \mathcal{K}^{\mathrm{coop}} \to \mathcal{K}$. For A, B in $\mathcal{K}$ we denote by $G_{A,B}$ the composite

$$\mathcal{K}(B, \widehat{A}) \cong \mathcal{K}(A^\circ \times B, \Omega) \xrightarrow{G_{\tau, A^\circ \times B}} \mathrm{DFib}(1, A^\circ \times B) \xrightarrow{d_{1,A,B}} \mathrm{DFib}(A, B)$$

Clearly the $G_{A,B}$ are fully faithful and pseudo-natural in A and B. A span

$$S : A \leftarrow E \rightarrow B$$

is an *attribute* when it is isomorphic to a span in the image of $G_{A,B}$. For $A \in \mathcal{K}$ we denote by

$$\epsilon_A : \ A \xleftarrow{\pi_A} \widehat{A}_\bullet \xrightarrow{u_A} \widehat{A}$$

the attribute $G_{A,\widehat{A}}(1_{\widehat{A}})$.

LEMMA 5.1. (1) *For any attribute $S : A \leftarrow E \rightarrow B$ there is an arrow $f : B \to \widehat{A}$ unique up to isomorphism such that $f^{\mathrm{rev}} \circ \epsilon_A \cong S$.*
(2) *Let $\alpha : A_1 \to A_2$, then $\widehat{\alpha}^{\mathrm{rev}} \circ \epsilon_{A_1} \cong \epsilon_{A_2} \circ \alpha$.*

PROOF. (1): If S is an attribute then by definition there is an $f : B \to \widehat{A}$ such that $G_{A,B}(f) \cong S$. By pseudo-naturality of the $G_{A,B}$ there is an isomorphism

$$\begin{array}{ccc} \mathcal{K}(\widehat{A}, \widehat{A}) & \xrightarrow{G_{A,\widehat{A}}} & \mathrm{DFib}(A, \widehat{A}) \\ {\scriptstyle -\circ f}\downarrow & \cong & \downarrow{\scriptstyle f^{\mathrm{rev}}\circ -} \\ \mathcal{K}(B, \widehat{A}) & \xrightarrow[G_{A,B}]{} & \mathrm{DFib}(A, B) \end{array}$$

whose component at $1_{\widehat{A}}$ is an isomorphism $G_{A,B}(f) \cong f^{\mathrm{rev}} \circ \epsilon_A$. If $g : B \to \widehat{A}$ also satisfies $g^{\mathrm{rev}} \circ \epsilon_A \cong S$, then we have $G_{A,B}(f) \cong G_{A,B}(g) \cong S$ and so $f \cong g$ by the fully faithfulness of $G_{A,B}$.
(2): The desired isomorphism arises by (1) and the component of $1_{\widehat{A}_2}$ of

$$\begin{array}{ccc} \mathcal{K}(\widehat{A}_2, \widehat{A}_2) & \xrightarrow{G_{A_2,\widehat{A}_2}} & \mathrm{DFib}(A_2, \widehat{A}_2) \\ {\scriptstyle \widehat{\alpha}\circ -}\downarrow & \cong & \downarrow{\scriptstyle -\circ\alpha} \\ \mathcal{K}(\widehat{A}_2, \widehat{A}_1) & \xrightarrow[G_{A_1,\widehat{A}_2}]{} & \mathrm{DFib}(A_1, \widehat{A}_2) \end{array}$$

$\square$

Another way to express lemma(5.1)(1) which is worth keeping in mind is that $G_{A,B}$ "is" span composition with ϵ_A. More precisely we have $G_{A,B} \cong (-)^{\mathrm{rev}} \circ \epsilon_A$. Define a map $f : A \to B$ to be admissible when the span f/B is an attribute. Thus there is an arrow $B(f,1) : B \to \widehat{A}$ unique up to isomorphism with the property that $f/B \cong B(f,1)^{\mathrm{rev}} \circ \epsilon_A$. The admissible arrows form a right ideal: given $f : A \to B$ admissible and $g : C \to B$ we have the following isomorphisms of spans

$$\begin{aligned} (fg)/B &\cong f/B \circ g \cong B(f,1)^{\mathrm{rev}} \circ \epsilon_A \circ g \\ &\cong B(f,1)^{\mathrm{rev}} \circ \widehat{g}^{\mathrm{rev}} \circ \epsilon_C \cong (\widehat{g}B(f,1))^{\mathrm{rev}} \circ \epsilon_C \end{aligned}$$

and so fg is admissible and $B(fg,1) \cong \widehat{g}B(f,1)$. In particular for an admissible object A we denote $A(1,1)$ as $y_A : A \to \widehat{A}$.

Proposition 5.2. *If $S \in \mathrm{Span}(A,B)$ is an attribute, A is an admissible object and $f : B \to \widehat{A}$ such that $S \cong f^{\mathrm{rev}} \circ \epsilon_A$; then $S \cong y_A/f$.*

Proof. By the definition of lax pullback it suffices to prove that for any span $S' : A \xleftarrow{d} E \xrightarrow{c} B$ there is a bijection between maps $S \to S'$ of spans and 2-cells $y_A d \to fc$, and that this bijection is natural in E. Noting that $S' \cong c \circ d^{\mathrm{rev}}$ and $c \dashv c^{\mathrm{rev}}$, maps $S \to S'$ are in bijection with maps $d^{\mathrm{rev}} \to c^{\mathrm{rev}} \circ S$. By the coyoneda lemma (theorem(2.12)) such maps are in bijection with $A/d \to c^{\mathrm{rev}} \circ S$ in $\mathrm{Span}(A,B)$. Now

$$A/d \cong d^{\mathrm{rev}} \circ A/A \cong d^{\mathrm{rev}} \circ {y_A}^{\mathrm{rev}} \circ \epsilon_A \cong (y_A d)^{\mathrm{rev}} \circ \epsilon_A$$

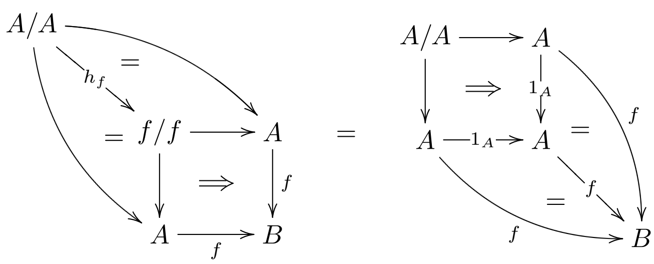
and

$$c^{\mathrm{rev}} \circ S \cong c^{\mathrm{rev}} f^{\mathrm{rev}} \circ \epsilon_A \cong (fc)^{\mathrm{rev}} \circ \epsilon_A$$

so we have a bijection between span maps $S' \to S$ and span maps $(y_A d)^{\mathrm{rev}} \circ \epsilon_A \to (fc)^{\mathrm{rev}} \circ \epsilon_A$. Since $G_{A,B} \cong (-)^{\mathrm{rev}} \circ \epsilon_A$ is fully faithful, the result follows. □

Thus when A is admissible, the map y_A is also admissible: by proposition(5.2) we have $\epsilon_A \cong y_A/\widehat{A}$ which is an attribute by definition, and so we have $\widehat{A}(y_A,1) \cong 1_{\widehat{A}}$.

To complete the specification of the data for a yoneda structure we must define for $f : A \to B$ with A and f admissible, a 2-cell $\chi^f : y_A \to B(f,1)f$. For such an f write φ_f for the isomorphism $B^{\mathrm{rev}}(f,1) \circ \epsilon_A \cong f/B$. Define $h_f : A/A \to f/f$ as the unique arrow such that



where the 2-cells in the above diagram are lax pullback cells. Then χ^f is defined as the unique 2-cell such that

(3)
$$\begin{array}{ccc} {y_A}^{\mathrm{rev}} \circ \epsilon_A & \xrightarrow{(\chi^f)^{\mathrm{rev}} \circ \epsilon_A} & (B(f,1)f)^{\mathrm{rev}} \circ \epsilon_A \cong f^{\mathrm{rev}} \circ B(f,1)^{\mathrm{rev}} \circ \epsilon_A \\ {\scriptstyle \varphi_{1_{\widehat{A}}}}\downarrow & & \downarrow{\scriptstyle f^{\mathrm{rev}} \circ \varphi_f} \\ A/A & \xrightarrow[h_f]{} & f/f \end{array}$$

commutes.

THEOREM 5.3. [**Str74a**] *Let A and $f : A \to B$ be admissible, and define χ^f as above.*

(1) *χ^f exhibits f as an absolute left lifting of y_A along $B(f,1)$.*
(2) *If*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle y_A}\searrow \overset{\phi}{\Rightarrow} \swarrow{\scriptstyle g} & \\ & \widehat{A} & \end{array}$$

which exhibits f as an absolute left lifting of y_A along g; then ϕ exhibits g as a pointwise left extension of y_A along f.

PROOF. (1): For any span $A \xleftarrow{a} X \xrightarrow{b} B$ we must exhibit a bijection between 2-cells $y_A a \to B(f,1)b$ and 2-cells $fa \to b$ natural in X which identifies χ^f and 1_f. Applying $G_{X,A}$ gives a bijection between 2-cells $y_A a \to B(f,1)b$ and maps $A/a \to f/b$ of spans. By the coyoneda lemma such span maps are in bijection with spans maps $a^{\mathrm{rev}} \to f/b$ and thus with 2-cells $fa \to b$ by the definition of f/b. This bijection is clearly natural in X, and tracing χ^f through these bijections clearly produces 1_f.
(2): The proof proceeds in two stages: first we show that the statement is true for $\phi = \chi^f$, and then in the general case we show that $g \cong B(f,1)$. For $b : X \to B$ we must show that pasting with

(4)
$$\begin{array}{ccc} f/b & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \\ & {\scriptstyle y_A}\searrow \overset{\chi^f}{\Rightarrow} \swarrow{\scriptstyle B(f,1)} & \\ & \widehat{A} & \end{array}$$

gives a bijection between 2-cells $B(f,1)b \to h$ and 2-cells $y_A p \to hq$. It suffices to exhibit a bijection between such 2-cells which is natural in X and which sends $1_{B(f,1)b}$ to the above composite. Applying $G_{X,A}$ and noting that $(B(f,1)b)^{\mathrm{rev}} \circ \epsilon_A \cong q \circ p^{\mathrm{rev}}$ exhibits a bijection between 2-cells $B(f,1)b \to h$ and span maps $q \circ p^{\mathrm{rev}} \to h^{\mathrm{rev}} \circ \epsilon_A$, and these are in bijection with span maps $p^{\mathrm{rev}} \to q^{\mathrm{rev}} \circ h^{\mathrm{rev}} \circ \epsilon_A$ since $q \dashv q^{\mathrm{rev}}$. By the coyoneda lemma and since $q^{\mathrm{rev}} \circ h^{\mathrm{rev}} \cong (hq)^{\mathrm{rev}}$, these are in turn in bijection with span maps $A/p \to (hq)^{\mathrm{rev}} \circ \epsilon_A$. Since $A/p \cong (y_A p)^{\mathrm{rev}} \circ \epsilon_A$, $G_{A,B} \cong (-)^{\mathrm{rev}} \circ \epsilon_A$ gives a bijection between such maps and with 2-cells $y_A p \to hq$ as required. These correspondences are clearly natural and map $1_{B(f,1)b}$ to the composite in (4). For the general case it now suffices to show $g \circ \epsilon_A \cong f/B$. It suffices to exhibit, for each span $S : A \xleftarrow{d} E \xrightarrow{c} B$, a bijection between span maps $S \to g \circ \epsilon_A$ and 2-cells $fd \to c$ natural in E. Composition ϕ gives a bijection between 2-cells $fd \to c$ and 2-cells $y_A d \to gc$ since ϕ is an absolute left lift. Applying $(-)^{\mathrm{rev}} \circ \epsilon_A$ and noting that $A/d \cong (y_A d)^{\mathrm{rev}} \circ \epsilon_A$ gives a bijection between 2-cells $y_A d \to gc$ and maps of spans $A/d \to c^{\mathrm{rev}} \circ g^{\mathrm{rev}} \circ \epsilon_A$, which by $c \dashv c^{\mathrm{rev}}$ correspond to maps of spans $S \to g^{\mathrm{rev}} \circ \epsilon_A$ since $S \cong c \circ d^{\mathrm{rev}}$. $\square$

This last result ensures that the axioms for the intended yoneda structure are indeed satisfied. We have proved:

Corollary 5.4. *Let $(\mathcal{K}, (-)^{\circ}, \tau)$ be a 2-topos. Then*

- $\widehat{A} = [A^{\circ}, \Omega]$.
- *$f : A \to B$ is admissible when f/B is an attribute.*
- *When A is admissible $y_A = A(1,1)$.*
- *For $f : A \to B$ such that A and f are admissible define $\chi^f : y_A \to B(f,1)f$ as in (3) above.*

specifies a good yoneda structure for $\mathcal{K}$.

6. Characterising admissible maps

When working with an example one would usually like to characterise what the admissible maps are. The following theorem is useful for this task. Given a 2-topos with classifying discrete opfibration τ, we shall say that a map $f : A \to B$ is *τ-admissible* when it is admissible in the sense of the yoneda structure on $\mathcal{K}$ induced by corollary(5.4). Similarly an object $A \in \mathcal{K}$ is *τ-small* when it is small in the sense of this induced yoneda structure. We shall also say that a map $f : A \to B$ in a 2-category factors *essentially through* a map $g : C \to B$ when there is a map $h : A \to C$ and an isomorphism $f \cong gh$.

Theorem 6.1. *Let $\mathcal{K}$ be a finitely complete cartesian closed 2-category with duality involution $(-)^{\circ}$. Suppose that τ and τ' are classifying discrete opfibrations fitting in a pullback square:*

$$\begin{array}{ccc} \Omega_{\bullet} & \longrightarrow & \Omega'_{\bullet} \\ {\scriptstyle \tau}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau'} \\ \Omega & \xrightarrow[i]{} & \Omega' \end{array}$$

(see proposition(4.4)). For $f : A \to B$ we use the notation introduced above (in sections(3) and (5)): $\widehat{A}$, $\widehat{A}_{\bullet}$, ϵ_A and (when f is τ-admissible) $B(f,1)$; for these aspects of the 2-topos $(\mathcal{K}, (-)^{\circ}, \tau)$. We use the notation $\widehat{A}'$, $\widehat{A}'_{\bullet}$, ϵ'_A and (when f is τ'-admissible) $B'(f,1)$; for the corresponding aspects of the 2-topos $(\mathcal{K}, (-)^{\circ}, \tau')$. If $f : A \to B$ is a τ'-admissible map then the following statements are equivalent.

1. *The adjoint transpose $A^{\circ} \times B \to \Omega'$ of $B'(f,1)$ factors essentially through i.*
2. *$B'(f,1)$ factors essentially through $[A^{\circ}, i]$.*
3. *f is τ-admissible.*

PROOF. (1)⇔(2) by the naturality of $\mathcal{K}$'s cartesian closed structure. In the diagram

$$\begin{array}{ccccc}
\mathcal{K}([A^\circ,\Omega'],[A^\circ,\Omega']) & \xrightarrow{\mathcal{K}([A^\circ,i],1)} & \mathcal{K}([A^\circ,\Omega],[A^\circ,\Omega']) & \xleftarrow{\mathcal{K}(1,[A^\circ,i])} & \mathcal{K}([A^\circ,\Omega],[A^\circ,\Omega]) \\
\downarrow & = & \downarrow & = & \downarrow \\
\mathcal{K}(A^\circ\times[A^\circ,\Omega'],\Omega') & \longrightarrow & \mathcal{K}(A^\circ\times[A^\circ,\Omega],\Omega') & \longleftarrow & \mathcal{K}(A^\circ\times[A^\circ,\Omega],\Omega) \\
{\scriptstyle G_{\tau'}}\downarrow & \cong & \searrow{\scriptstyle G_{\tau'}} & \cong & \downarrow{\scriptstyle G_\tau} \\
\mathrm{DFib}(1,A^\circ\times[A^\circ,\Omega']) & & \xrightarrow{(A^\circ\times[A^\circ,i])^{\mathrm{rev}}\circ(-)} & & \mathrm{DFib}(1,A^\circ\times[A^\circ,\Omega]) \\
\downarrow & & \cong & & \downarrow \\
\mathrm{DFib}(A,[A^\circ,\Omega']) & & \xrightarrow[{[A^\circ,i]^{\mathrm{rev}}\circ(-)}]{} & & \mathrm{DFib}(A,[A^\circ,\Omega])
\end{array}$$

the top row of vertical arrows are the isomorphisms coming from $\mathcal{K}$'s cartesian closed structure, and the bottom row of vertical arrows are the equivalences coming from the duality involution. The top two squares commute by the naturality of the cartesian closed structure. The triangle in the middle commutes up to isomorphism by the definition of $G_{\tau'}$ and G_τ (we saw this also in proposition(4.4)). The other isomorphisms are pseudo naturality isomorphisms. Tracing $1_{\widehat{A'}}$ and $1_{\widehat{A}}$ through this diagram, and recalling the definitions of ϵ'_A and ϵ_A, we obtain an isomorphism of spans

$$\epsilon_A \cong [A^\circ,i]^{\mathrm{rev}} \circ \epsilon'_A.$$

Suppose that $B'(f,1)$ factors essentially through $[A^\circ,i]$, so that there is a map $h : B \to \widehat{A}$ such that $B'(f,1) \cong [A^\circ,i]h$. Then

$$f/B \cong h^{\mathrm{rev}} \circ [A^\circ,i]^{\mathrm{rev}} \circ \epsilon'_A \cong f/B \cong h^{\mathrm{rev}} \circ \epsilon_A$$

whence f is τ admissible and $h \cong B(f,1)$. Conversely if $f : A \to B$ is τ-admissible then by definition there is $B(f,1) : B \to \widehat{A}$ such that $f/B \cong B(f,1)^{\mathrm{rev}} \circ \epsilon_A$. Thus $[A^\circ,i]B(f,1) \cong B'(f,1)$ by lemma(5.1). □

EXAMPLE 6.2. Taking CAT with its usual duality involution, and τ to be the forgetful functor $U : \mathrm{Set}_\bullet \to \mathrm{Set}$, corollary(5.4) produces the yoneda structure foreshadowed in example(3.3). For $A \in \mathrm{CAT}$, $\widehat{A}_\bullet$ is the following category of "pointed presheaves":

- objects: are triples (x, F, a) where $F \in \widehat{A}$, $a \in A$, and $x \in F(a)$.
- arrows: an arrow $(x_1, F_1, a_1) \to (x_2, F_2, a_2)$ is a pair (ϕ, α) where $\phi : F_1 \to F_2$ in $\widehat{A}$ and $\alpha : a_1 \to a_2$ in A such that $\phi_a(x_1) = F_2(\alpha)(x_2)$.

and

$$A \xleftarrow{\pi_A} \widehat{A}_\bullet \xrightarrow{U_A} \widehat{A}$$

are the obvious forgetful functors. We stated in example(3.3) that $f : A \to B$ is admissible iff $B(fa, b)$ is a small set for all $a \in A$ and $b \in B$. To see this is the case let CAT' be a cartesian closed 2-category of categories which contains SET as an object and CAT as a sub-2-category. In the situation of theorem(6.1) with $\mathcal{K} = \mathrm{CAT}'$, $\Omega' = \mathrm{SET}$, $\Omega = \mathrm{Set}$ and $i : \mathrm{Set} \to \mathrm{SET}$ the inclusion, notice that any

$f : A \to B$ in CAT is τ'-admissible because $B'(f, 1)$ can be defined as the adjoint transpose of

$$A^{\mathrm{op}} \times B \to \mathrm{SET}$$

given on objects by $(a, b) \mapsto B(fa, b)$. It is straight forward to observe directly that one has an isomorphism of spans $f/B \cong B(f,1)^{\mathrm{rev}} \circ \epsilon'_A$ and that there is a pullback square

$$\begin{array}{ccc} \widehat{A}_{\bullet} & \longrightarrow & \widehat{A}'_{\bullet} \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle U'_A} \\ \widehat{A} & \xrightarrow[{[A^{\mathrm{op}}, i]}]{} & \widehat{A}' \end{array}$$

in CAT$'$. Thus by theorem(6.1) f is admissible iff $\forall a, b$, $B(fa, b)$ is in Set as claimed in example(3.3). In particular $A \in$ CAT is admissible iff its hom-sets are small. Clearly a category A equivalent to one with a small set of arrows will be small in our sense: A and $\widehat{A}$ are admissible. The converse, that A and $\widehat{A}$ admissible implies that A is equivalent to a category which has a small set of arrows, is a result of Freyd and Street [**FS95**].

Remark 6.3. The results of [**FS95**] apply also when "small" is taken to mean finite, or more generally "of cardinality less than λ" where λ is a regular cardinal. Thus throughout this work, one could replace Set by the category of sets of cardinality less than λ.

Example 6.4. Here is an example of a 2-topos whose presheaves are *not* cocomplete. Consider the classifying discrete opfibration obtained from $U : \mathrm{Set}_{\bullet} \to \mathrm{Set}$ by pulling it back along the inclusion of the full subcategory of Set consisting of the non-empty sets. By theorem(6.1) a category is admissible for the resulting yoneda structure on CAT when its hom-sets are non-empty and small. The empty category 0 is admissible (vacuously), and so is $1 \cong \widehat{0}$, so that 0 is small. Thus initial objects are small colimits for this yoneda structure. However the category of non-empty sets lacks an initial object.

Example 6.5. In this example we characterise admissible maps and small objects for example(4.7) where $\mathcal{K} = \mathrm{CAT}(\widehat{\mathbb{C}})$ and $\Omega = \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$. Unpacking the definitions and the duality involution one obtains that for $A \in \mathrm{CAT}(\widehat{\mathbb{C}})$ and $C \in \mathbb{C}$, $\widehat{A}_{\bullet}(C)$ is the category:

- objects: are triples (x, F, a) where $F \in \widehat{A}(C)$, $a \in AC$, and $x \in F(a)(1_C)$.
- arrows: an arrow $(x_1, F_1, a_1) \to (x_2, F_2, a_2)$ is a pair (ϕ, α) where $\phi : F_1 \to F_2$ in $\widehat{A}(C)$ and $\alpha : a_1 \to a_2$ in AC such that $\phi_{a,1_C}(x_1) = F_2(\alpha)_{1_C}(x_2)$.

To characterise the admissible maps we use theorem(6.1) with $\mathcal{K} = [C^{\mathrm{op}}, \mathrm{CAT}']$, $\Omega' = \mathrm{Sp}_{\mathbb{C}}(\mathrm{SET})$ via a version of example(4.7) with CAT$'$ in place of CAT, and $\Omega = \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$. As before notice that any $f : A \to B$ in $\mathrm{CAT}(\widehat{\mathbb{C}})$ is τ'-admissible, and that the pullback square relating τ and τ' can be obtained by applying $\mathrm{Sp}_{\mathbb{C}}$ to the corresponding pullback square found in the previous example. Thus by theorem(6.1) it follows that f is admissible iff $\forall C \in \mathbb{C}, a \in AC, b \in BC$, the set $B(C)(fa, b)$ is small. Thus in particular $A \in \mathrm{CAT}(\widehat{\mathbb{C}})$ is admissible iff $\forall C$, $A(C)$ has small hom-sets. If for all C, $A(C)$ is equivalent to a category with a small set of arrows, then since $\mathbb{C}$ is small, A is small in the sense that A and $\widehat{A}$ are admissible. For the converse note that

$$\begin{array}{rcl} \widehat{A}(C) & \cong & \mathrm{CAT}(\widehat{\mathbb{C}})(C, [A^\circ, \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})]) \cong \mathrm{CAT}(\widehat{\mathbb{C}})(C \times A^\circ, \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})) \\ & \cong & \mathrm{CAT}(\mathrm{el}(C \times A)^{\mathrm{op}}, \mathrm{Set}) \end{array}$$

and so if $\widehat{A}(C)$ is locally small for all C, then by [**FS95**] $\mathrm{el}(C \times A)$ is small for all C, and this implies that $A(C)$ is small for all C.

Example 6.6. In this example we characterise the admissible maps and small objects for example(4.9) where $\mathcal{K} = \mathrm{CAT}(\widehat{\mathbb{G}})$ and $\Omega = \Sigma^k \mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$. We shall now write $\tau : \Omega_\bullet \to \Omega$ for the classifying discrete opfibration $\mathrm{Sp}_{\mathbb{G}}(U)$ in $\mathrm{CAT}(\widehat{\mathbb{G}})$, where $U : \mathrm{Set}_\bullet \to \mathrm{Set}$ is the forgetful functor. We saw that Σ preserves classifying discrete opfibrations. By the characterisation of the τ-admissible arrows and τ-small objects given in the previous example, Σ clearly preserves τ-admissible arrows and τ-small objects. Noting that Σy_1 is τ-admissible and writing i_1 for the map $\Sigma\Omega(\Sigma y_1, 1)$, we have

$$\begin{array}{ccc} 1 & \xrightarrow{\Sigma y_1} & \Sigma\Omega \\ & {\scriptstyle y_1} \searrow \quad \overset{\chi^{\Sigma y_1}}{\Rightarrow} \quad \swarrow {\scriptstyle i_1} & \\ & \Omega & \end{array}$$

exhibiting i_1 as a pointwise left extension of y_1 along Σy_1, and Σy_1 as an absolute left lifting of y_1 along i_1[10]. By proposition(5.2) there is a 2-cell

$$\begin{array}{ccc} \Omega_\bullet & \xrightarrow{\tau} & \Omega \\ \downarrow & \overset{\lambda}{\Rightarrow} & \downarrow {\scriptstyle 1} \\ 1 & \xrightarrow[y_1]{} & \Omega \end{array}$$

which exhibits $\Omega_\bullet$ as y_1/Ω and since Σ is a right adjoint it preserves this lax pullback. Define λ' as follows:

$$\begin{array}{ccc} \Sigma\Omega_\bullet & \xrightarrow{\Sigma\tau} & \Sigma\Omega \\ \downarrow & \overset{\lambda'}{\Rightarrow} & \downarrow {\scriptstyle i_1} \\ 1 & \xrightarrow[y_1]{} & \Omega \end{array} \quad = \quad \begin{array}{ccc} \Sigma\Omega_\bullet & \xrightarrow{\Sigma\tau} & \Sigma\Omega \\ \downarrow & \overset{\Sigma\lambda}{\Rightarrow} & \downarrow {\scriptstyle 1} \\ 1 & \xrightarrow{\Sigma y_1} & \Sigma\Omega \\ & {\scriptstyle y_1} \searrow \quad \overset{\chi^{\Sigma y_1}}{\Rightarrow} \quad \swarrow {\scriptstyle i_1} & \\ & \Omega & \end{array}$$

Since $\Sigma\lambda$ is a lax pullback and $\chi^{\Sigma y_1}$ is an absolute left lifting, it follows easily that λ' exhibits $\Sigma\Omega_\bullet$ as y_1/i_1. By the universal property for λ, there is a unique $i_{1,\bullet}$

[10]Since 1 is τ-admissible we saw in example(8.3) that $t_\Omega \dashv y_1$ and so y_1 is a fully faithful right adjoint. Fully faithful right adjoints are preserved by all 2-functors and so Σy_1 is fully faithful, whence $\chi^{\Sigma y_1}$ is in fact invertible by proposition(2.22).

fitting in a commutative square

(5)
$$\begin{array}{ccc} \Sigma\Omega_{\bullet} & \xrightarrow{i_{1,\bullet}} & \Omega_{\bullet} \\ {\scriptstyle \Sigma\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ \Sigma\Omega & \xrightarrow[i_1]{} & \Omega \end{array}$$

such that $\lambda i_{1,\bullet} = \lambda'$, and since λ' is a lax pullback, (5) is a pullback. Denoting $\mathcal{K} = [\mathbb{G}^{\mathrm{op}}, \mathrm{CAT}']$, $\Omega' = \mathrm{Sp}_{\mathbb{G}}(\mathrm{SET})$ and τ' for the corresponding classifying discrete opfibration in $\mathcal{K}$, we have

$$\begin{array}{ccccc} \Sigma\Omega_{\bullet} & \longrightarrow & \Omega_{\bullet} & \longrightarrow & \Omega'_{\bullet} \\ {\scriptstyle \Sigma\tau}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} & \lrcorner & \downarrow{\scriptstyle \tau'} \\ \Sigma\Omega & \xrightarrow[i_1]{} & \Omega & \xrightarrow[i]{} & \Omega' \end{array}$$

in $\mathcal{K}$, and as before, every $f : A{\rightarrow}B$ in $\mathrm{CAT}(\widehat{\mathbb{G}})$ is τ'-admissible as a map in $\mathcal{K}$. By theorem(6.1) f is $\Sigma\tau$-admissible iff it is τ-admissible and the adjoint transpose of $B(f,1)$ factors through i_1. This last condition says that $\forall a \in A_0, b \in B_0$, the set $B_0(fa, b)$ is a singleton. In particular, $A \in \mathrm{CAT}(\widehat{\mathbb{G}})$ is $\Sigma\tau$-admissible when A_0 is chaotic[11] and the A_n are locally small for $n > 0$. Using the left 2-adjoint D of Σ described in example(4.9), we can say these things a little more efficiently. A map $f : A{\rightarrow}B$ of $\mathrm{CAT}(\widehat{\mathbb{G}})$ is $\Sigma\tau$-admissible iff Df is τ-admissible and $\forall a \in A_0, b \in B_0$, the set $B_0(fa, b)$ is a singleton. An object A of $\mathrm{CAT}(\widehat{\mathbb{G}})$ is $\Sigma\tau$-admissible iff DA is τ-admissible and A_0 is chaotic. By the adjointness $D \dashv \Sigma$, cartesian closedness, and since D preserves products and commutes with the duality involution we have isomorphisms

$$\mathrm{CAT}(\widehat{\mathbb{G}})(X, [A^{\circ}, \Sigma\Omega]) \cong \mathrm{CAT}(\widehat{\mathbb{G}})(DX \times DA^{\circ}, \Omega) \cong \mathrm{CAT}(\widehat{\mathbb{G}})(X, \Sigma\widehat{DA})$$

natural in X, and so by the 2-categorical yoneda lemma, we get an isomorphism $[A^{\circ}, \Sigma\Omega] \cong \Sigma\widehat{DA}$. Thus, $A \in \mathrm{CAT}(\widehat{\mathbb{G}})$ is $\Sigma\tau$-small iff DA and $\widehat{DA}$ are τ-admissible and A_0 is chaotic, that is, iff DA is τ-small and A_0 is chaotic.

Let $k \in \mathbb{N}$. Then replacing Σ by Σ^k, and thus D by the left adjoint D^k of Σ^k in the above analysis, one can characterise the $\Sigma^k\tau$-admissible maps and objects, and the $\Sigma^k\tau$-small objects in exactly the same way. The results of this are:

(1) $f : A{\rightarrow}B$ is $\Sigma^k\tau$-admissible iff for $n < k$: $\forall a \in A_n, b \in B_n$ there is a unique arrow $fa{\rightarrow}b$ in B_n; and for $n{\geq}k$: $\forall a \in A_n, b \in B_n$, $B_n(fa, b)$ is in Set.

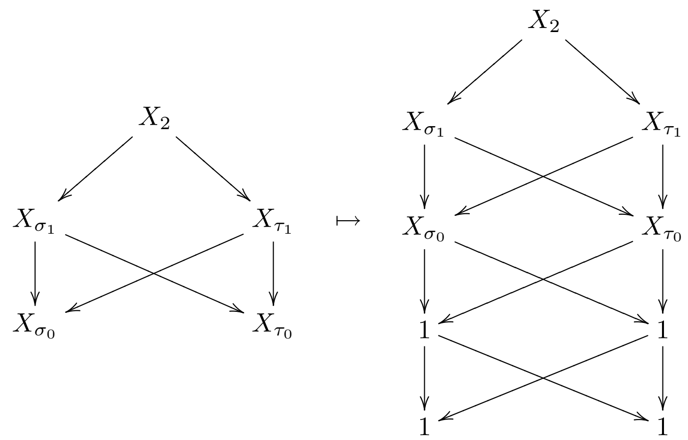
(2) A is $\Sigma^k\tau$-admissible iff for $n < k$ the A_n are chaotic and for $n{\geq}k$ the A_n are locally small.

(3) A is $\Sigma^k\tau$-small iff for $n < k$ the A_n are chaotic and for $n{\geq}k$ the A_n are small.

Analogously to the case $k = 1$ we denote the map $\Sigma^k\Omega(\Sigma^k y_1, 1)$ by i_k. The reader may verify, by unpacking the necessary definitions, that i_k has the following explicit description. For $n{\geq}k$ an object of $\Sigma^k\Omega_n$ is an $(n-k)$-span of small sets. The effect of i_k on such an $(n-k)$-span X is the n-span obtained by putting 1 (a terminal

[11]Recall that a category X is *chaotic* when $\forall x, y \in X$ there is a unique arrow $x{\rightarrow}y$ in X. Thus a chaotic category is either empty or equivalent to 1.

object of Set) in the bottom k-levels and X in the top $(n-k)$-levels. For example for $k=2$ and $n=4$:



and so i_k is an "internalisation" of Σ^k. In fact $\mathrm{CAT}(\widehat{\mathbb{G}})(1, i_k)$ is exactly the endofunctor of $\widehat{\mathbb{G}}$ given by right kan extension along $(-+1)$ mentioned in example(4.9).

7. Cocomplete presheaves for 2-toposes

This section is devoted to explaining why for our main examples of 2-toposes, the resulting yoneda structures do indeed have cocomplete presheaf objects. The main result involves providing conditions on a 2-adjunction between 2-categories equipped with good yoneda structures, which ensure that the right 2-adjoint preserves small cocomplete and cocomplete objects.

While the left 2-adjoints in these situations do not preserve lax pullbacks, they come close to doing so in a way that is useful to us. Recall that a *left adjoint retraction* for an arrow $f : A \to B$ in a 2-category $\mathcal{K}$ is an adjunction $l \dashv f$ in $\mathcal{K}$ such that the counit is an identity. Similarly a right adjoint retraction for f is an adjunction $f \dashv r$ whose unit is an identity.

Definition 7.1. A 2-functor $F : \mathcal{A} \to \mathcal{B}$ between finitely complete 2-categories *preserves lax pullbacks up to a right adjoint section* when for all

$$W \xrightarrow{f} Z \xleftarrow{g} X$$

in $\mathcal{A}$, the canonical comparison map $F(f/g) \to Ff/Fg$ has a left adjoint retraction over FX.

To spell this definition out a little more, note that by definition the comparison map $F(f/g) \to Ff/Fg$ may be regarded as an arrow in $\mathcal{B}/FW$ (over FW) and also as an arrow in $\mathcal{B}/FX$ (over FX). To say that the comparison has a left adjoint retraction over FX is the same as saying that is has a left adjoint retraction in $\mathcal{B}/FX$. The dual definition, that F preserves lax pullbacks up to a *left* adjoint section, says that the comparison has a right adjoint retraction in $\mathcal{B}/FW$. Obviously any 2-functor which actually preserves lax pullbacks does so in both these weaker senses.

Example 7.2. The 2-functor E discussed in example(4.7) preserves lax pullbacks up to a right adjoint section. To prove this, given the definition of E, it suffices to show that $\mathrm{el} : \mathrm{CAT}(\widehat{\mathbb{C}}) \to \mathrm{CAT}$ preserves lax pullbacks up to a left adjoint section. Consider

$$W \xrightarrow{f} Z \xleftarrow{g} X$$

in $\mathrm{CAT}(\widehat{\mathbb{C}})$ and write $k : \mathrm{el}(f/g) \to \mathrm{el}(f)/\mathrm{el}(g)$ for the comparison map. An explicit description of the category $\mathrm{el}(f)/\mathrm{el}(g)$ is:

- objects may be regarded as 6-tuples $(C_1, w, p, \phi, x, C_2)$ where

$$\begin{array}{ccccc} C_1 & & \xrightarrow{p} & & C_2 \\ {\scriptstyle w}\downarrow & & \overset{\phi}{\Longrightarrow} & & \downarrow{\scriptstyle x} \\ W & \xrightarrow[f]{} & Z & \xleftarrow[g]{} & X \end{array}$$

  is in $\mathrm{CAT}(\widehat{\mathbb{C}})$ and the $C_i \in \mathbb{C}$ are regarded as representables in the usual way.
- a map

$$(C_1, w, p, \phi, x, C_2) \to (C'_1, w', p', \phi', x', C'_2)$$

  consists of a 4-tuple (a, α, β, b) where

$$\begin{array}{ccc} C_1 & \xrightarrow{a} & C'_1 \\ {\scriptstyle w}\searrow & \overset{\alpha}{\Rightarrow} & \swarrow{\scriptstyle w'} \\ & W & \end{array} \qquad \begin{array}{ccc} C_2 & \xrightarrow{b} & C'_2 \\ {\scriptstyle x}\searrow & \overset{\beta}{\Rightarrow} & \swarrow{\scriptstyle x'} \\ & Z & \end{array}$$

  is in $\mathrm{CAT}(\widehat{\mathbb{C}})$ such that $p'a = bp$ and $(\phi' a)(f\alpha) = (g\beta p)\phi$.

The comparison k may be regarded as the full inclusion of those $(C_1, w, p, \phi, x, C_2)$ such that $p = \mathrm{id}$. The right adjoint retraction r for k is given by

$$r(C_1, w, p, \phi, x, C_2) = (C_1, w, \mathrm{id}, \phi, xp, C_1) \qquad r(a, \alpha, \beta, b) = (a, \alpha, \beta p, a)$$

and the counit ε has component at $(C_1, w, p, \phi, x, C_2)$ given by $(1_{C_1}, \mathrm{id}, \mathrm{id}, p)$. Clearly r is over $\mathrm{el}(W)$, $\varepsilon k = \mathrm{id}$ and $r\varepsilon = \mathrm{id}$, and so ε is indeed the counit for an adjunction $k \dashv r$ over $\mathrm{el}(W)$ whose unit is an identity.

EXAMPLE 7.3. Let $\mathcal{K}$ be a finitely complete 2-category and $Z \in \mathcal{K}$. Then $(-)\times Z : \mathcal{K} \to \mathcal{K}$ preserves lax pullbacks up to a left and right adjoint section. To see this first recall the map $i : Z \to Z/Z$ whose composite with the lax pullback

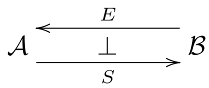
$$\begin{array}{ccc} Z/Z & \xrightarrow{q_Z} & Z \\ {\scriptstyle p_Z}\downarrow & \overset{\lambda_Z}{\Longrightarrow} & \downarrow{\scriptstyle 1} \\ Z & \xrightarrow[1]{} & Z \end{array}$$

is the identity on 1_Z. Defining $\eta : 1_{Z/Z} \to iq_Z$ as the unique 2-cell such that $p_Z\eta = \lambda_Z$ and $q_Z\eta = \mathrm{id}$, it is easily verified that η is the unit of an adjunction over Z with counit an identity. In more detail one has an adjunction in $\mathcal{K}/Z$

$$1_Z \underset{i}{\overset{q_Z}{\leftrightarrows}} p_Z \ (\bot) \qquad\qquad 1_Z \underset{p_Z}{\overset{i}{\rightleftarrows}} q_Z \ (\bot)$$

as displayed on the left with counit an identity, and dually an adjunction in $\mathcal{K}/Z$ as displayed on the right with unit an identity. Now there is a canonical isomorphism $f/g\times Z \cong (f\times Z)/(g\times Z)$ whose composite with the comparison map is $f/g\times i$: this is easy verified directly in CAT, and so by a representable argument happens in $\mathcal{K}$. Taking the cartesian product of f/g with the above adjunctions in $\mathcal{K}/Z$ gives the desired adjoints.

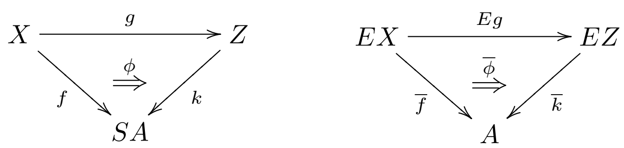
THEOREM 7.4. *Let*



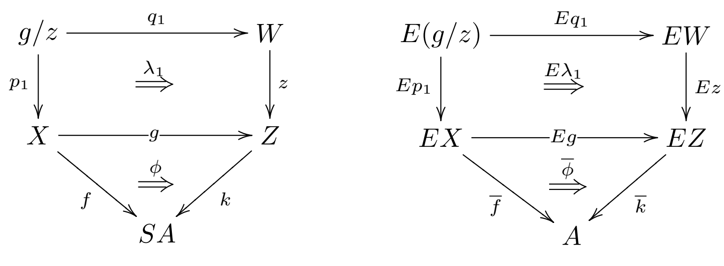
be a 2*-adjunction,* $\mathcal{A}$ *and* $\mathcal{B}$ *be finitely complete* 2*-categories each equipped with a good yoneda structure which exhibits small lax pullbacks. Suppose that* S *preserves admissible objects and* E *preserves lax pullbacks up to a right adjoint section. Then*

(1) *If* E *preserves small objects then* S *preserves objects which are both admissible and small cocomplete.*
(2) *If* E *also preserves admissible objects then* S *preserves objects which are both admissible and cocomplete.*

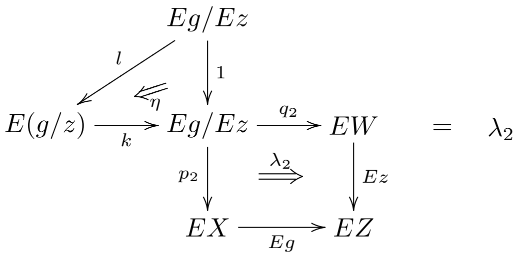
PROOF. Since the yoneda structures on $\mathcal{A}$ and $\mathcal{B}$ admit small pullbacks, (small) cocompleteness for admissible objects may be expressed purely in terms of pointwise left extensions by corollary(3.16) and definition(3.17). To prove (1) we consider $g : X \to Z$ and $f : X \to SA$ in $\mathcal{B}$, where A is admissible and small cocomplete, and X and Z are small. We must demonstrate the existence of a pointwise left extension of f along g. Define ϕ and k



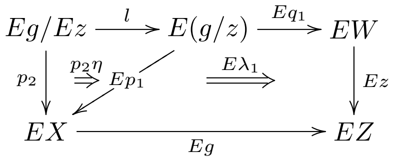
to be the mates of $\overline{\phi}$ and $\overline{k}$ under $E \dashv S$, where $\overline{\phi}$ exhibits $\overline{k}$ as a pointwise left extension of $\overline{f}$, the mate of f, along Eg. Mateship of such 2-cells clearly preserves left extensions as it is compatible with pasting of 2-cells $k \to k'$, so ϕ is clearly a left extension, but we must demonstrate that this left extension is in fact pointwise. To this end let $z : W \to Z$. We must show that the composite 2-cell on the left



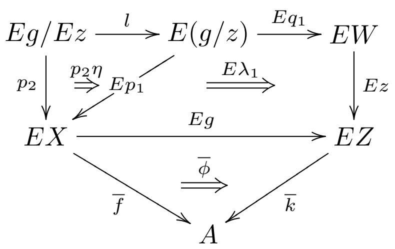
is a left extension, and this is equivalent to saying that its mate, the composite on the right, is a left extension. Write $k : E(g/z) \to Eg/Ez$ for the comparison, l for its left adjoint and η for the unit of $l \dashv k$. Writing λ_2 for the defining lax pullback of Eg/Ez, we have



since $q_2\eta = \mathrm{id}$, and so λ_2 factors as



Since η is the unit of an adjunction, $p_2\eta$ exhibits Ep_1 as an absolute left extension of p_2 along l. The result now follows by the elementary properties of left extensions since the composite



and $\overline{f}p_2\eta$ are left extensions. The proof of (2) is identical to (1) except that Z is only assumed to be admissible. $\square$

Example 7.5. Let $\mathbb{C}$ be a small category and consider the 2-topos structure on $\mathrm{CAT}(\widehat{\mathbb{C}})$ we began to analyze in example(6.5), where $\Omega = \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$. We will now show that the resulting yoneda structure has cocomplete presheaf objects.

For a small object $A \in \mathrm{CAT}(\widehat{\mathbb{C}})$ the 2-adjunction

$$\mathrm{CAT}(\widehat{\mathbb{C}}) \underset{[A,-]}{\overset{(-)\times A}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{CAT}(\widehat{\mathbb{C}})$$

satisfies all the hypotheses of theorem(7.4). The hypotheses of theorem(7.4) concerned with admissibility or smallness follow easily from the characterisation of small and admissible objects given in example(6.5), and $(-)\times A$ preserves lax pullbacks up to a right adjoint section by example(7.3).

Thus if A is small and X is admissible and cocomplete, then $[A, X]$ is admissible and cocomplete.

In particular to see that $\mathrm{CAT}(\widehat{\mathbb{C}})$ has cocomplete presheaf objects it suffices to exhibit Ω as cocomplete. We now consider the 2-adjunction

$$\mathrm{CAT} \underset{\mathrm{Sp}_{\mathbb{C}}}{\overset{E}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{CAT}(\widehat{\mathbb{C}})$$

discussed in example(4.7), and regard CAT as a 2-topos with classifying discrete opfibration $U : \mathrm{Set}_{\bullet} \to \mathrm{Set}$. By the characterisation of admissible and small objects given in examples(6.2) and (6.5), and by example(7.2), this adjunction satisfies the conditions of theorem(7.4). Thus if $\mathcal{E} \in \mathrm{CAT}$ is locally small and cocomplete, then $\mathrm{Sp}_{\mathbb{C}}(\mathcal{E})$ is admissible and cocomplete.

In the case $\mathbb{C} = \mathbb{G}$ a corollary of presheaf cocompleteness and theorem(3.20) is that the globular category of higher spans of sets is the colimit completion of 1.

Example 7.6. We now continue the analysis of example(6.6) and adopt the notation used there. Moreover we adopt the following terminology: an object $A \in \mathrm{CAT}(\widehat{\mathbb{G}})$ is $\Sigma^k\tau$*-cocomplete* when it is cocomplete for the yoneda structure arising from the 2-topos $(\mathrm{CAT}(\widehat{\mathbb{G}}), (-)^{\circ}, \Sigma^k\tau)$ by theorem(5.4). We shall now see that this yoneda structure also has cocomplete presheaf objects.

The hypotheses of theorem(7.4) are trivially satisfied for the 2-adjunction $D \dashv \Sigma$, where the yoneda structure for both $\mathcal{A}$ and $\mathcal{B}$ is that induced by τ. Thus if $A \in \mathrm{CAT}(\widehat{\mathbb{G}})$ is τ-admissible and τ-cocomplete then so is $\Sigma^k A$ for $k \in \mathbb{N}$. In particular $\Sigma^k\Omega$ is τ-cocomplete, and so is $[A^{\circ}, \Sigma^k\Omega]$ when A is τ-small by the previous example. From the characterisation of smalls and admissibles given in example(6.6), τ-admissible implies $\Sigma^k\tau$-admissible and τ-cocomplete implies $\Sigma^k\tau$-cocomplete.

8. The cartesian closedness of Ω

One of the facets of 2-topos theory is that Ω, the codomain of a classifying discrete opfibration inherits quite a bit of natural structure. A full discussion of this involves describing how the theory of cartesian 2-monads interacts with 2-topos theory, and is provided in [**Web**]. We shall now describe how Ω is cartesian closed.

Definition 8.1. In a finitely complete 2-category $\mathcal{K}$, an object A is a *cartesian pseudo monoid* when the maps

$$1 \longleftarrow A \xrightarrow{\Delta} A \times A$$

have right adjoints. If for each $a : 1 \to A$ the map $(- \times a)$, defined as the composite

$$A \cong A \times 1 \xrightarrow{1_A \times a} A \times A \xrightarrow{m} A$$

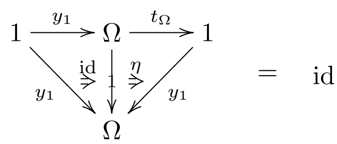
where $\Delta \dashv m$, has a right adjoint, then A is said to be *cartesian closed*.

In CAT a cartesian pseudo monoid is a category with finite products, and an object of CAT is cartesian closed in the sense of definition(8.1) when it is a cartesian closed category in the usual sense. We now fix a 2-topos $(\mathcal{K}, (-)^{\circ}, \tau)$ and under some hypotheses clearly satisfied by our main examples[12], exhibit Ω as a cartesian closed object of $\mathcal{K}$.

[12] In particular examples(4.2), (4.7), and (4.9) and the analogues of these obtained by replacing Set everywhere by Set_{λ} for any regular cardinal λ.

Proposition 8.2. *If* 1 *is admissible then* $y_1 : 1 \to \Omega$ *is right adjoint to* t_Ω*.*

Proof. Define η by



so that η exhibits y_1 as a left extension along t_Ω. Note that this left extension is preserved by t_Ω and so $t_\Omega \dashv y_1$. □

Example 8.3. We now exhibit an example of a 2-topos for which 1 is not admissible. Consider the classifying discrete opfibration obtained from $U : \mathrm{Set}_\bullet \to \mathrm{Set}$ by pulling it back along the inclusion of the full subcategory of Set consisting of those sets of cardinality at least 2. For the resulting yoneda structure on CAT, 1 is not admissible since this full subcategory of Set lacks a terminal object.

The following result exhibits the multiplication of a cartesian pseudo monoid structure on Ω in 2-topos theoretic terms. A discrete fibration in $\mathcal{K}$ has *small fibres* when it is an attribute. Given a discrete fibration (d_1, E_1, c_1) from A_1 to B_1 and a discrete fibration (d_2, E_2, c_2) from A_2 to B_2, then $(d_1 \times d_2, E_1 \times E_2, c_1 \times c_2)$ is a discrete fibration from $A_1 \times A_2$ to $B_1 \times B_2$: this is easily seen to be true in CAT by direct inspection, and then in general by the representability of the notions involved. Thus in particular $\tau \times \tau$ is a discrete opfibration.

Proposition 8.4. *Suppose that* $\tau \times \tau$ *has small fibres. Then a map*

$$m : \Omega \times \Omega \to \Omega$$

which classifies $\tau \times \tau$ *is right adjoint to the diagonal map* Δ_Ω*.*

Proof. Note that $\Omega_\bullet$ may be identified with the head of the span y_1/Ω by proposition(5.2). For $x, z_1, z_2 : X \to \Omega$ we must exhibit a bijection between 2-cells $x \to m(z_1, z_2)$ and $x\Delta_\Omega \to (z_1, z_2)$ which is natural in X and in (z_1, z_2): then the 2-cell $\eta : 1 \to m\Delta_\Omega$ corresponding to the identity exhibits Δ_Ω as an absolute left lifting of 1_Ω along m, and so is the unit of an adjunction. Maps $x \to m(z_1, z_2)$ are in bijection with maps of spans $y_1/x \to y_1/m(z_1, z_2)$ by the universal property of τ. Now $m(z_1, z_2) = m(z_1 \times z_2)\Delta_X$ so that $y_1/m(z_1, z_2) \cong \Delta_X^{\mathrm{rev}} \circ y_1/m(z_1 \times z_2)$, and so by the adjointness $\Delta_X \dashv \Delta_X^{\mathrm{rev}}$, span maps $y_1/x \to y_1/m(z_1, z_2)$ are in bijection with span maps $\Delta_X \circ y_1/x \to y_1/m(z_1 \times z_2)$. From the pullbacks

$$\begin{array}{ccccc} y_1/z_1 \times y_1/z_2 & \longrightarrow & y_1/\Omega \times y_1/\Omega & \longrightarrow & y_1/\Omega \\ \downarrow & & \downarrow{\scriptstyle \tau \times \tau} & & \downarrow{\scriptstyle \tau} \\ X \times X & \xrightarrow[z_1 \times z_2]{} & \Omega \times \Omega & \xrightarrow[m]{} & \Omega \end{array}$$

one has $y_1/m(z_1 \times z_2) \cong y_1/z_1 \times y_1/z_2$, and so span maps $\Delta_X \circ y_1/x \to y_1/m(z_1 \times z_2)$ are in bijection with span maps $\Delta_X \circ y_1/x \to y_1/z_1 \times y_1/z_2$, which are in bijection with pairs of span maps $(y_1/x \to y_1/z_1, y_1/x \to y_1/z_2)$, and these are in bijection with pairs of 2-cells $(x \to z_1, x \to z_2)$ by the universal property of τ. Finally these pairs of 2-cells are in bijection with 2-cells $x\Delta_\Omega \to (z_1, z_2)$. It is straightforward to verify that these bijections are easily are respected by composition with maps into X, and by composition with 2-cells out of (z_1, z_2). □

In the main examples the hypotheses of these results are easy to verify, but on the other hand, the fact that Ω is a cartesian pseudo monoid is easily verifiable without them: cartesian pseudo monoids are preserved by finite product preserving 2-functors, and our main examples are obtained from well-known examples in CAT by applying certain right 2-adjoints.

However cartesian closed objects are not preserved by right 2-adjoints[13] and so proposition(8.4) is useful to us because it specifies *how* the cartesian pseudo monoid structure of Ω is specified in 2-topos theoretic terms. We exploit this in the proof of theorem(8.6). First we require a lemma, which is an analogue of the yoneda lemma for 2-sided discrete fibrations (theorem(2.12)).

LEMMA 8.5. *Let $\mathcal{K}$ be a finitely complete 2-category and $f : A \to B$ be in $\mathcal{K}$. Recall the definition of the map $i_f : A \to f/B$ prior to theorem(2.12). For any span (d_1, E_1, c_1) from X to Z and discrete fibration (d_2, E_2, c_2) from $A \times X$ to $B \times Z$, composition with*

$$i_f \times \mathrm{id} : f \times E_1 \to f/B \times E_1$$

determines a bijection between maps of spans $f/B \times E_1 \to E_2$ and maps of spans $f \times E_1 \to E_2$.

PROOF. By the representability of the notions involved it suffices to prove this result for the case $\mathcal{K} = \mathrm{CAT}$. In this case the head of the span $f/B \times E_1$ can be described as follows:

- objects are 4-tuples $(e, a, \beta : fa \to b, b)$ where $e \in E_1$.
- an arrow
$$(\varepsilon, \alpha, \beta) : (e_1, a_1, \beta_1, b_1) \to (e_2, a_2, \beta_2, b_2)$$
consists of maps $\varepsilon : e_1 \to e_2$, $\alpha : a_1 \to a_2$ and $\beta : b_1 \to b_2$, such that $\beta\beta_1 = \beta_2 f(\alpha)$.

and the left and right legs of the span send $(\varepsilon, \alpha, \beta)$ described above to $(\alpha, d_1\varepsilon)$ and $(\beta, c_1\varepsilon)$ respectively. The image of $i_f \times \mathrm{id}$ is the full subcategory given by the (e, a, β, b) such that $\beta = \mathrm{id}$. Let $\phi : f/B \times E_1 \to E_2$. For any object (e, a, β, b) we have a map

$$(1, 1, \beta) : (e, a, \mathrm{id}, fa) \to (e, a, \beta, b)$$

and since $d_2\phi(1, 1, \beta) = \mathrm{id}$ and $c_2\phi(1, 1, \beta) = (\beta, \mathrm{id})$, $\phi(e, a, \beta, b)$ and $\phi(1, 1, \beta)$ are defined uniquely by $(\phi(e, a, 1_{fa}, fa), \beta)$ since E_2 is a discrete fibration. For any arrow $(\varepsilon, \alpha, \beta)$ as above, we have a commutative square

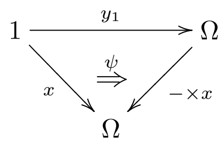
$$\begin{array}{ccc} \phi(e_1, a_1, \mathrm{id}, fa_1) & \xrightarrow{\phi(1,1,\beta_1)} & \phi(e_1, a_1, \beta_1, b_1) \\ {\scriptstyle \phi(\varepsilon,\alpha,f\alpha)}\downarrow & & \downarrow{\scriptstyle \phi(\varepsilon,\alpha,\beta)} \\ \phi(e_2, a_2, \mathrm{id}, fa_2) & \xrightarrow[\phi(1,1,\beta_2)]{} & \phi(e_2, a_2, \beta_2, b_2) \end{array}$$

in E_2, but from the proof of theorem(2.11) we know that $\phi(1, 1, \beta_1)$ is d_2-opcartesian and so the rest of the above square is determined uniquely by $\phi(\varepsilon, \alpha, f\alpha)$. □

[13]For an example note that the functor category $[\Sigma\mathbb{N}, \mathrm{Set}_f]$ is not cartesian closed where $\Sigma\mathbb{N}$ is the monoid of natural numbers under addition regarded as a one object category, and Set_f is the category of finite sets. Thus the representable 2-functor $\mathrm{CAT}(\Sigma\mathbb{N}, -)$ doesn't preserve cartesian closed objects.

THEOREM 8.6. *If* 1 *is small,* Ω *is cocomplete and* $\tau \times \tau$ *has small fibres, then* Ω *is cartesian closed.*

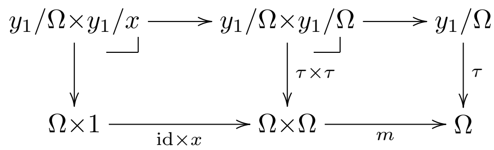
PROOF. For $x : 1 \to \Omega$ it suffices to provide



which exhibits $-\times x$ as a left extension of x along y_1: since Ω is cocomplete, the left extension of x along y_1 exists and is left adjoint to $\Omega(x, 1)$ by theorem(3.20). We do this by exhibiting a bijection between 2-cells $x \to py_1$ and 2-cells $(-\times x) \to p$ which is natural in p. By the universal property of τ 2-cells $x \to py_1$ are in bijection with span maps $y_1/x \to y_1/py_1$. Since $y_1/py_1 \cong y_1^{\text{rev}} \circ y_1/p$ and $y_1 \dashv y_1^{\text{rev}}$, these span maps are in bijection with span maps $y_1 \circ y_1/x \to y_1/p$. We have span isomorphisms

$$y_1 \circ y_1/x \cong (y_1 \times \text{id}) \circ (\text{id} \times y_1/x) \cong y_1 \times y_1/x$$

and so span maps $y_1 \circ y_1/x \to y_1/p$ are in bijection with span maps $y_1 \times y_1/x \to y_1/p$ which by lemma(8.5) are in bijection with span maps $y_1/\Omega \times y_1/x \to y_1/p$. From the pullbacks



one has $y_1/\Omega \times y_1/x \cong y_1/m(\text{id} \times x)$, and so span maps $y_1/\Omega \times y_1/x \to y_1/p$ are in bijection with span maps $y_1/m(\text{id} \times x) \to y_1/p$, and these are in bijection with 2-cells $(-\times x) \to p$ by the universal property of τ. It is straight forward to verify that the bijections just described are natural in p. □

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