Algebras of higher operads as enriched categories

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To the memory of Max Kelly

ABSTRACT. One of the open problems in higher category theory is the systematic construction of the higher dimensional analogues of the Gray tensor product. In this paper we begin to adapt the machinery of globular operads [**1**] to this task. We present a general construction of a tensor product on the category of n-globular sets from any normalised $(n+1)$-operad A, in such a way that the algebras for A may be recaptured as enriched categories for the induced tensor product. This is an important step in reconciling the globular and simplicial approaches to higher category theory, because in the simplicial approaches one proceeds inductively following the idea that a weak $(n+1)$-category is something like a category enriched in weak n-categories. In this paper we reveal how such an intuition may be formulated in terms of globular operads.

1. Introduction

The subject of enriched category theory [**9**] and 2-category theory was brought to maturity by the efforts of Max Kelly and his collaborators. Max also had a hand in the genesis of the study of operads, and in [**10**] which for a long time went unpublished, he laid the categorical basis for their further analysis. It is with great pleasure that we are able to present the following paper, which relates enriched category theory and the study of higher operads, in dedication to a great mathematician.

Coherence issues for higher dimensional categories become interesting from dimension 3. In dimension 2 as is well-known, any bicategory is biequivalent to a strict 2-category. However for tricategories, there is something in between strict and weak – namely *Gray categories*. These are obtained by defining a symmetric monoidal closed structure on 2-Cat, the category of strict 2-categories and 2-functors between them. This Gray tensor product is subtly different from the cartesian product of 2-categories, and Gray-categories are categories enriched in 2-Cat using the Gray tensor product. The coherence theorem for tricategories [**8**] then says that any tricategory is triequivalent to a Gray category. The analogous statement, using instead strict 3-categories, is false. With strict 3-categories definable as categories enriched in 2-Cat using the *cartesian* tensor product, it becomes apparent that understanding tensor products of higher categories is fundamental for coherence.

In [**6**] Crans attempted to push our understanding one dimension higher by defining a tensor product of Gray categories, the ultimate hope being that weak

4-categories could be shown to be equivalent in an appropriate sense to categories enriched in Gray-Cat using this new tensor product. However technical difficulties abound with this approach. The tensor product in [6] is described explicitly, and this is extremely complicated. Clearly understanding the situation in still higher dimensions requires new ideas. Moreover this Crans tensor product is not biclosed, and so the proposed semi-strict 4-categories are not enriched in themselves. From experience with enriched category theory and its uses in the study of Gray-categories, this is a serious obstruction to the development of the theory. However in discussions with the first named author, Crans seemed to have obtained a proposal for subtly altering his definition so that the resulting tensor product is biclosed, but unfortunately this was never published.

The situation is very similar to that just before the combinatorial definition of [1] of weak ω-category as given. At that time some people had tried to give a combinatorial definition of weak 4-category, but such attempts were extremely complicated. The time was ripe for a new idea and [1] provided it, and in so doing gave a language which enables one to deal with all the complexity inherent to defining higher categories combinatorially. From our present perspective the idea is the following. The description of any n-dimensional categorical structure X may begin by starting with just the underlying globular data, that is, sets and functions

$$X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} X_3 \underset{t}{\overset{s}{\leftleftarrows}} \cdots \underset{t}{\overset{s}{\leftleftarrows}} X_n$$

satisfying the equations $ss = st$ and $ts = tt$. The elements of X_0 are the objects, the elements of X_1 are the arrows and so on, and such a complex is called an n-globular set. However at this stage no compositions have been defined, and when they are, one has a globular set with extra structure. *Thus the problem of defining an n-categorical structure of a given type is that of defining the monad on the category $\widehat{\mathbb{G}}_{\leq n}$ of n-globular sets whose algebras are these structures.* The surprise is that this is actually feasible, and in fact leads to a conceptual theory, because apriori one might expect that the explicit description of these monads would be formidable.

However there are important reasons why they are not. First, the monad $\mathcal{T}_{\leq n}$ for strict n-categories can be described explicitly in an appealing way using trees as was done in [1]. Second, this monad has very good formal properties – it is a *local right adjoint monad* in the sense to be recalled below – and in [15] it was shown that such monads on presheaf categories are in fact easy to handle. In fact for every such monad T, one can define the nerve of a T-algebra which satisfies many properties that generalise the simplicial nerve of a category. Third, the monads that define weaker n-categorical structures all come equipped with cartesian monad morphisms into $\mathcal{T}_{\leq n}$. Thus these monads are themselves also local right adjoints, and moreover, we understand a lot about them combinatorially because we understand $\mathcal{T}_{\leq n}$. In fact one can define an n-operad as exactly this: a cartesian monad morphism $\alpha : A \to \mathcal{T}_{\leq n}$ [1], and an algebra for it is just an algebra of A. All n-dimensional categorical structures of interest arise as algebras of n-operads, and in [1], weak n-categories were defined by constructing the appropriate n-operad.

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[1]That is, A is another monad on $\widehat{\mathbb{G}}_{\leq n}$, α consists of a natural transformation between the underlying functors of A and $\mathcal{T}$, and this is compatible with the monad units and multiplications, and moreover the naturality squares of α are pullbacks. In fact, by the results of section(6), compatibility with the monad structure is a consequence of the cartesianness of α, and moreover given A, such an α is unique if it exists.

In this paper we begin to adapt higher operads to the task of defining tensor products in higher category theory. An operad or monad on $\widehat{\mathbb{G}}_{\leq n}$ is normalised when it doesn't affect the objects. All n-dimensional categorical structures of interest are in fact algebras of normalised n-operads. To each normalised $(n + 1)$-operad A, we shall define an associated tensor product on the category $\widehat{\mathbb{G}}_{\leq n}$. This tensor product is *lax*, meaning that the coherence data for it involves non-invertible arrows. Nevertheless, one can still define categories enriched in a lax monoidal category, and one recovers the algebras of A as the categories enriched in the associated tensor product. This is just the beginning. For in the case of the 3-operad for Gray categories, the induced tensor product is on the category $\widehat{\mathbb{G}}_{\leq 2}$, whereas the Gray tensor product lives on 2-Cat. In the sequel [**3**] to this paper, we shall explain how to lift the tensor products constructed here to the category of algebras of an appropriate operad, and in this way, recover the Gray tensor product and many other examples including that of Crans [**6**].

This paper is organised as follows. In section(2) we recall the definition of a lax monoidal category and of categories enriched therein. Multitensors, that is lax monoidal structures, generalise non-symmetric operads, and sections(3) and (4) explain how basic operad theory generalises to multitensors. In section(3) we see how under certain conditions, one may regard multitensors as monoids for a certain monoidal structure, which generalises the substitution tensor product of collections familiar from the theory of operads. Proposition(3.3) is in fact a special case of proposition(2.1) of [**7**]. Nevertheless we give a self-contained account of proposition(3.3) and related notions, to keep the exposition relatively self-contained and as elementary as possible for our purposes. In section(4) we explain how one can induce a monad from a multitensor. The theory of T-multitensors, which is the multitensorial analogue of the theory of T-operads for a cartesian monad T described in [**12**], is given in section(5). If T preserves coproducts, then it distributes over the monoid monad which we denote as M, and we obtain an equivalence between $\mathcal{T}$-multitensors and $M\mathcal{T}$-operads.

From this point in the paper we begin working directly with the case $T = \mathcal{T}$, the strict ω-category monad on the category $\widehat{\mathbb{G}}$ of globular sets. In section(6) we give a self-contained inductive description of the monad $\mathcal{T}$ and develop further some of its properties. The inductive description of $\mathcal{T}$ given here is closely related to the wreath product of Clemens Berger [**4**]. In section(7) we give the correspondence between normalised $\mathcal{T}$-operads and $\mathcal{T}$-multitensors, as well as the identification between the algebras of a given normalised operad and categories enriched in the associated multitensor. In the final section we explain how our results may be adapted to normalised n-operads, that is to finite dimensions, and then explain how the algebras of $\mathcal{T}$, which we defined as a combinatorial object, really are strict ω-categories. This last fact is of course well-known, but the simplicity and canonicity of our proof is a pleasant illustration of the theory developed in this paper.

2. Lax Monoidal Categories

In this section we recall the notion of *lax monoidal category*, which is a generalisation of the well-known concept of monoidal category. As with monoidal categories, one can consider categories enriched in a lax monoidal category. Any

monad T on a finitely complete category $\mathcal{V}$ defines a canonical lax monoidal structure $T^{\times}$ on $\mathcal{V}$, and for this structure enriched categories correspond to categories enriched in T-Alg regarded as monoidal via cartesian product.

Given a 2-monad T on a 2-category $\mathcal{K}$ one may consider *lax algebras* for T. A lax T-algebra structure on an object $A \in \mathcal{K}$ is a triple (a, u, σ) consisting of an action $a : TA \rightarrow A$ together with 2-cells

$$
\begin{array}{ccc}
A \xrightarrow{\eta_A} TA & \quad & T^2A \xrightarrow{\mu_A} TA \\
\quad\quad 1_A \searrow \underset{\Rightarrow}{u} \swarrow a & \quad & Ta \downarrow \quad \overset{\sigma}{\Rightarrow} \quad \downarrow a \\
\quad A & \quad & TA \xrightarrow{a} A
\end{array}
$$

satisfying some well-known axioms. See [**11**] for a complete description of these axioms, and of the 2-category Lax-T-Alg. When T is the identity, lax algebras are just monads in $\mathcal{K}$. The example most important for us however is when T is the monoid monad $\mathcal{M}$ on CAT.

DEFINITION 2.1. A *multitensor* on a category $\mathcal{V}$ is a lax $\mathcal{M}$-algebra structure (E, u, σ) on $\mathcal{V}$. A category $\mathcal{V}$ equipped with a multitensor structure is called a *lax monoidal category*. When u is the identity the multitensor and lax monoidal structure are said to be *normal*.

We shall now unpack this definition. Since $\mathcal{M}\mathcal{V} = \coprod_{n \geq 0} \mathcal{V}^n$ a functor

$$E : \mathcal{M}\mathcal{V} \rightarrow \mathcal{V}$$

amounts to functors $E_n : \mathcal{V}^n \rightarrow \mathcal{V}$ for $n \in \mathbb{N}$.

Before proceeding further we digress a little on notation. For functors of many variables we shall use some space saving notation: we deem that the following expressions

$$E_n(X_1, ..., X_n) \qquad \underset{1 \leq i \leq n}{\mathrm{E}} X_i \qquad \underset{i}{\mathrm{E}} X_i$$

are synonymous, and we will frequently use the latter, often leaving the "n" unmentioned when no confusion would result. In particular for $X \in \mathcal{V}$ and $1 \leq i \leq n$, $\underset{i}{\mathrm{E}} X$ denotes $E_n(X, ..., X)$. We identify the number n with the ordered set $\{1, ..., n\}$ and we refer to elements of the ordinal sum $n_{\bullet} := n_1 + ... + n_k$ as pairs (i, j) where $1 \leq i \leq k$ and $1 \leq j \leq n_i$. Following these conventions $\underset{i}{\mathrm{E}} \underset{j}{\mathrm{E}} X_{ij}$ and $\underset{ij}{\mathrm{E}} X_{ij}$ are synonymous with

$$E_k(E_{n_1}(X_{11}, ..., X_{1n_1}), ..., E_{n_k}(X_{k1}, ..., X_{kn_k}))$$

and

$$E_{n_\bullet}(X_{11}, ..., X_{1n_1}, ..., X_{k1}, ..., X_{kn_k})$$

respectively. We will use multiply indexed expressions (like $\underset{i}{\mathrm{E}} \underset{jk}{\mathrm{E}} \underset{l}{\mathrm{E}} X_{ijkl}$) to more efficiently convey expressions that have multiple layers of brackets and applications of E's.

The remaining data for a multitensor on $\mathcal{V}$ amounts to maps

$$u_X : X \rightarrow E_1 X \qquad \sigma_{X_{ij}} : \underset{i}{\mathrm{E}} \underset{j}{\mathrm{E}} X_{ij} \rightarrow \underset{ij}{\mathrm{E}} X_{ij}$$

that are natural in the arguments and satisfy

$$
\begin{array}{ccc}
\underset{i}{\mathrm{E}}\, X_i \xrightarrow{\,u\,\mathrm{E}\,i\,} E_1 \underset{i}{\mathrm{E}}\, X_i & \underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{E}}\,\underset{k}{\mathrm{E}}\, X_{ijk} \xrightarrow{\,\sigma\,\underset{k}{\mathrm{E}}\,} \underset{ij}{\mathrm{E}}\,\underset{k}{\mathrm{E}}\, X_{ijk} & \underset{i}{\mathrm{E}}\, E_1 X_i \xleftarrow{\,\mathrm{E}\,u\,i\,} \underset{i}{\mathrm{E}}\, X_i
\end{array}
$$

Thus a multitensor is very much like a functor-operad in the sense of [13], except that there are no symmetric group actions with respect to which the substitutions are equivariant[2]. An equivalent formulation of definition(2.1), in the language of [2], is that a multitensor on $\mathcal{V}$ is a non-symmetric operad internal to the endomorphism operad of $\mathcal{V}$.

EXAMPLE 2.2. A normal multitensor on $\mathcal{V}$ such that σ is invertible is just a monoidal structure on $\mathcal{V}$, with E_n playing the role of the n-fold tensor product. In the case where $\mathcal{V}$ is finitely complete and E_n is n-fold cartesian product and for the sake of the next example, we denote the isomorphism "σ" as

$$
\iota : \prod_i \prod_j X_{ij} \to \prod_{ij} X_{ij}
$$

EXAMPLE 2.3. Let T be a monad on a finitely complete category $\mathcal{V}$. Denote by

$$
k_{X_i} : T \prod_i X_i \to \prod_i T X_i
$$

the canonical maps which measure the extent to which T preserves products. One defines a multitensor $(T^\times, u, \sigma)$ as follows:

$$
T_k^\times(X_1, ..., X_k) = \prod_{1 \le i \le n} T(X_i)
$$

u is the unit $\eta_X : X \to TX$ of the monad, and σ is defined as the composite

$$
\prod_i T \prod_j T X_{ij} \xrightarrow{\prod_i kT} \prod_i \prod_j T^2 X_{ij} \xrightarrow{\iota\mu} \prod_{ij} T X_{ij}
$$

For the remainder of this section let $(\mathcal{V}, E)$ be a lax monoidal category.

DEFINITION 2.4. An *E-category* (X, κ), or in other words a category enriched in $(\mathcal{V}, E)$, consists of

- a set X_0 of objects.
- for all pairs (x_0, x_1) of elements of X_0, an object $X(x_0, x_1)$ of $\mathcal{V}$. These objects are called the *homs* of X.
- for all $n \in \mathbb{N}$ and $(n+1)$-tuples $(x_0, ..., x_n)$ of elements of X_0, maps

$$
\kappa_{x_i} : \underset{1 \le i \le n}{\mathrm{E}}\, X(x_{i-1}, x_i) \to X(x_0, x_n)
$$

called the *compositions* of X.

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[2]More precisely, functor-operads in the sense of [13] are normal lax algebras for the *symmetric* monoidal category 2-monad on CAT.

satisfying unit and associative laws, which say that

$$X(x_0, x_1) \xrightarrow{u} E_1 X(x_0, x_1) \qquad \underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{E}}\, X(x_{(ij)-1}, x_{ij}) \xrightarrow{\sigma} \underset{ij}{\mathrm{E}}\, X(x_{(ij)-1}, x_{ij})$$

commute, where $1 \leq i \leq m$, $1 \leq j \leq n_i$ and $x_{(11)-1} = x_0$. Since a choice of i and j references an element of the ordinal $n_\bullet$, the predecessor $(ij)-1$ of the pair (ij) is well-defined when i and j are not both 1. An *E-monoid* is an *E*-category with one object.

DEFINITION 2.5. Let (X, κ) and (Y, λ) be *E*-categories. An *E-functor* $f : (X, \kappa) \to (Y, \lambda)$ consists of a function $f_0 : X_0 \to Y_0$, and for all pairs (x_0, x_1) from X_0, arrows

$$f_{x_0, x_1} : X(x_0, x_1) \to Y(fx_0, fx_1)$$

satisfying a functoriality axiom, which says that

$$
\begin{array}{ccc}
\underset{i}{\mathrm{E}}\, X(x_{i-1}, x_i) & \xrightarrow{\underset{i}{\mathrm{E}}\, f} & \underset{i}{\mathrm{E}}\, Y(fx_{i-1}, fx_i) \\
{\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle \lambda} \\
X(x_0, x_n) & \xrightarrow{f} & Y(fx_0, fx_n)
\end{array}
$$

commutes. We denote by *E*-Cat the category of *E*-categories and *E*-functors, and by $\mathrm{Mon}(E)$ the full subcategory of *E*-Cat consisting of the *E*-monoids.

EXAMPLE 2.6. A non-symmetric operad

$$(A_n \ : \ n \in \mathbb{N}) \quad u : I \to A_1 \quad \sigma : A_k \otimes A_{n_1} \otimes ... \otimes A_{n_k} \to A_{n_\bullet}$$

in a braided monoidal category $\mathcal{V}$ defines a multitensor E on $\mathcal{V}$ via the formula

$$\underset{1 \leq i \leq n}{\mathrm{E}} X_i = A_n \otimes X_1 \otimes ... \otimes X_n$$

with u and σ providing the structure maps in the obvious way. The category $\mathrm{Mon}(E)$ of *E*-monoids is the usual category of algebras of A, and thus *E*-categories are a natural notion of "many object algebra" for an operad A.

Our notation for multitensors makes evident the analogy with monads and algebras: a multitensor E is analogous to a monad and an *E*-category is the analogue of an algebra for E. In particular observe that the following basic facts are instances of the axioms for the lax monoidal category $(\mathcal{V}, E)$ and categories enriched therein.

LEMMA 2.7.
(1) (E_1, u, σ) *is a monad on* $\mathcal{V}$.
(2) *The monad* E_1 *acts on* E_n *for all* $n \in \mathbb{N}$, *that is*

$$\sigma : E_1 \underset{i}{\mathrm{E}} X_i \to \underset{i}{\mathrm{E}} X_i$$

is an E_1-*algebra structure on* $\underset{i}{\mathrm{E}} X_i$.
(3) *With respect to the* E_1-*algebra structures of (2) all of the components of* σ *are* E_1-*algebra morphisms.*

(4) *Each hom of an E-category (X, κ) is an E_1-algebra, with the algebra struc-*
ture on $X(x_0, x_1)$ given by

$$\kappa : E_1 X(x_0, x_1) \rightarrow X(x_0, x_1).$$

(5) *With respect to the E_1-algebras of (2) and (4), all the components of κ*
are morphisms of E_1-algebras.

PROPOSITION 2.8. *Let T be a monad on a finitely complete category $\mathcal{V}$. Re-*
garding T-Alg as a monoidal category via cartesian product one has

$$T^{\times}\text{-Cat} \cong (T\text{-Alg})\text{-Cat}$$

commuting with the forgetful functors into Set.

PROOF. Let X_0 be a set and for $a, b \in X_0$ let $X(a, b) \in \mathcal{V}$. Suppose that

$$\kappa_{x_i} : \prod_i TX(x_{i-1}, x_i) \rightarrow X(x_0, x_n)$$

for each $n \in \mathbb{N}$ and $x_0, ..., x_n$ in X_0, are the structure maps for a $T^{\times}$-category
structure. Then by lemma(2.7) the $\kappa_{a,b} : TX(a, b) \rightarrow X(a, b)$ are algebra structures
for the homs, and for $x_{ij} \in X_0$ with $1 \leq i \leq k$ and $1 \leq j \leq n_i$ one has the inner regions
of

commutative, and the commutative outer region is the associativity axiom for the
composites

$$\kappa'_{x_i} : \prod_i X(x_{i-1}, x_i) \xrightarrow{\prod_i \eta} \prod_i TX(x_{i-1}, x_i) \xrightarrow{\kappa_{x_i}} X(x_0, x_n)$$

for each $x_0, ..., x_n \in X_0$. Taking the product structure on T-Alg as normal, the
unit axiom for the κ' is clearly satisfied, and so they are the structure maps for a
$(T\text{-Alg})$-category structure. Conversely given algebra structures $\kappa_{a,b}$ and structure
maps κ'_{x_i} one can define κ_{x_i} as the composite

$$\prod_i TX(x_{i-1}, x_i) \xrightarrow{\prod_i \kappa_{x_{i-1}, x_i}} \prod_i X(x_{i-1}, x_i) \xrightarrow{\kappa'_{x_i}} X(x_0, x_n)$$

and since the regions of

$$\prod_i T \prod_j TX(x_{(ij)-1}, x_{ij}) \xrightarrow{\ k\ } \prod_{ij} T^2 X(x_{(ij)-1}, x_{ij}) \xrightarrow{\ \prod_{ij} \mu\ } \prod_{ij} TX(x_{(ij)-1}, x_{ij})$$

commute, the commutativity of the outside of this diagram shows that the κ_{x_i} satisfy the associativity condition of a $T^\times$-category structure, and the unit axiom follows from the unit T-algebra axiom on the homs. The correspondence just described is clearly a bijection, and completes the description of the isomorphism on objects over Set.

Let $f_0 : X_0 \rightarrow Y_0$ be a function,

$$\kappa_{x_i} : \prod_i TX(x_{i-1}, x_i) \rightarrow X(x_0, x_n) \qquad \lambda_{y_i} : \prod_i TY(y_{i-1}, y_i) \rightarrow Y(y_0, y_n)$$

be the structure maps for $T^\times$-categories X and Y, κ'_{x_i} and λ'_{y_i} be the associated $(T\text{-Alg})$-category structures, and

$$f_{a,b} : X(a, b) \rightarrow Y(fa, fb)$$

for $a, b \in X_0$ be maps in $\mathcal{V}$. In the following display the diagram on the left

explains how the $T^\times$-functor axiom for the $f_{a,b}$ implies the $(T\text{-Alg})$-functor axiom, and the diagram on the right shows the converse. $\square$

3. Distributive multitensors as monoids

It is well-known that monads on a category $\mathcal{V}$ are monoids in the strict monoidal category $\mathrm{End}(\mathcal{V})$ of endofunctors of $\mathcal{V}$ whose tensor product is given by composition. Given the analogy between monads and multitensors, one is led to ask under what circumstances are multitensors monoids in a certain monoidal category. One natural answer to this question, that we shall present now, requires that we restrict attention to *distributive* multitensors to be defined below. Throughout this section $\mathcal{V}$ is assumed to have coproducts.

DEFINITION 3.1. A functor $E : \mathcal{MV} \to \mathcal{V}$ is *distributive* when for all $n \in \mathbb{N}$, E_n preserves coproducts in each variable. We denote by $\mathrm{Dist}(\mathcal{V})$ the category whose objects are such functors $\mathcal{MV} \to \mathcal{V}$, and whose morphisms are natural transformations between them. A multitensor (E, u, σ) (resp. lax monoidal category $(\mathcal{V}, E)$) is said to be *distributive* when E is distributive.

EXAMPLES 3.2. In the case where $(\mathcal{V}, \otimes, I)$ is a genuine monoidal category, $\mathcal{V}$ is distributive in the above sense iff $(X \otimes -)$ and $(- \otimes X)$ preserve coproducts for each $X \in \mathcal{V}$. If in addition $\otimes$ is just cartesian product and T is a monad on $\mathcal{V}$ whose functor part preserves coproducts, then the multitensor $T^{\times}$ of example(2.3) is also distributive.

When E is distributive we have

$$\mathop{\mathrm{E}}_{1 \leq i \leq n} \coprod_{j \in J_i} X_{ij} \cong \coprod_{j_1 \in J_1} \dots \coprod_{j_n \in J_n} \mathop{\mathrm{E}}_i X_{ij_i}$$

for any doubly indexed family X_{ij} of objects of $\mathcal{V}$. To characterise distributivity via this formula we must be more precise and say that a certain canonical map between these objects is an isomorphism. It is however more convenient to express all this in terms of coproduct cocones. To state such an equation we must have for each $1 \leq i \leq n$ a family of maps

$$(c_{ij} : X_{ij} \to X_{i\bullet} : j \in J_i)$$

which forms a coproduct cocone in $\mathcal{V}$. Given a choice for each i of $j \in J_i$, one obtains a map

$$\mathop{\mathrm{E}}_i c_{ij} : \mathop{\mathrm{E}}_i X_{ij} \to \mathop{\mathrm{E}}_i X_{i\bullet},$$

and distributivity says that all such maps together form a coproduct cocone. The morphisms that comprise this cocone are indexed by elements of $\prod_i J_i$ in agreement with the right hand side of the above formula. For what will soon follow it is worth recalling that the (obviously true) statement "a coproduct of coproducts is a coproduct" can be described in a similar way. That is, given c_{ij} as above together with another coproduct cocone

$$(c_i : X_{i\bullet} \to X_{\bullet\bullet} : 1 \leq i \leq n),$$

for each choice of i and j one obtains a composite arrow

$$X_{ij} \xrightarrow{c_{ij}} X_{i\bullet} \xrightarrow{c_i} X_{\bullet\bullet} ,$$

and the collection of all such composites is a coproduct cocone.

Define the unit I of $\mathrm{Dist}(\mathcal{V})$ by $I_1 = 1_{\mathcal{V}}$ and for $n \neq 1$, I_n is constant at $\emptyset$. The tensor product $E \circ F$ of E and F in $\mathrm{Dist}(\mathcal{V})$ is defined as:

$$(E \circ F)_n = \coprod_{k \geq 0} \coprod_{n_1 + \dots + n_k = n} \mathop{\mathrm{E}}_i F_{n_i}$$

and so for all k and $n_i \in \mathbb{N}$ where $1 \leq i \leq k$ we have maps

$$\mathop{\mathrm{E}}_i \mathop{\mathrm{F}}_j \xrightarrow{\ \mathop{c}\limits_{ij}\ } \mathop{\mathrm{E} \circ \mathrm{F}}_{ij}$$

which we shall also denote by $c_{(n_1,\ldots,n_k)}$ as convenience dictates. For all $n \in \mathbb{N}$ the set of all such maps such that $n_\bullet = n$ form a coproduct cocone. In the case where $E = I$ one has $\underset{i}{I} \underset{j}{F} \cong \emptyset$ when $k \neq 1$, and so

$$c_{(n)} : F_n \to (I \circ F)_n$$

is invertible, the inverse of which we denote by λ. In the case where $F = I$ one has $\underset{i}{E} \underset{j}{I} \cong \emptyset$ when not all the n_i's are 1, and so

$$c_{(1,\ldots,1)} : E_n \to (E \circ 1)_n$$

is invertible, the inverse of which we denote by ρ. Given E, F and G in $\mathrm{Dist}(\mathcal{V})$, one has for all $r \in \mathbb{N}$, $m_i \in \mathbb{N}$ such that $1 \leq i \leq r$, and $n_{ij} \in \mathbb{N}$ for all i and $1 \leq j \leq m_i$, a composite

$$\underset{i}{E} \underset{j}{F} \underset{k}{G} \xrightarrow{\underset{ij}{c}\ \underset{k}{G}} (\underset{ij}{E \circ F}) \underset{k}{G} \xrightarrow{\underset{(ij)k}{c}} ((E \circ F) \circ G)_{n_{\bullet\bullet}}$$

and for all $n \in \mathbb{N}$, the set of all such composites obtained from such choices with $n_{\bullet\bullet} = n$ forms a coproduct cocone (the coproduct of coproducts is a coproduct). For a given choice of r, m_i and n_{ij} as above one can also form a composite

$$\underset{i}{E} \underset{j}{F} \underset{k}{G} \xrightarrow{\underset{i}{E}\ \underset{jk}{c}} \underset{i}{E}(\underset{jk}{F \circ G}) \xrightarrow{\underset{i(jk)}{c}} (E \circ (F \circ G))_{n_{\bullet\bullet}}$$

and for all $n \in \mathbb{N}$, the set of all such composites obtained from such choices with $n_{\bullet\bullet} = n$ forms a coproduct cocone because E is distributive. Thus for each n, E, F and G one has a unique isomorphism α, such that for all choices of r, m_i and n_{ij} with $n_{\bullet\bullet} = n$, the diagram

(1)
$$\begin{array}{ccc}
\underset{i}{E} \underset{j}{F} \underset{k}{G} & \xrightarrow{\underset{ij}{c}\ \underset{k}{G}} (\underset{ij}{E \circ F}) \underset{k}{G} \xrightarrow{\underset{(ij)k}{c}} & ((E \circ F) \circ G)_n \\
& \searrow^{\underset{i}{E}\ \underset{jk}{c}} & \downarrow^{\alpha} \\
& \underset{i}{E}(\underset{jk}{F \circ G}) \xrightarrow[\underset{i(jk)}{c}]{} & (E \circ (F \circ G))_n
\end{array}$$

commutes.

PROPOSITION 3.3. *The data $(I, \circ, \alpha, \lambda, \rho)$ just described is a monoidal structure for $\mathrm{Dist}(\mathcal{V})$. The category $\mathrm{Mon}(\mathrm{Dist}(\mathcal{V}))$ is isomorphic to the category of distributive multitensors and morphisms thereof.*

PROOF. The case of (1) for which $m_i = 1$ amounts to the commutativity of the outside of

$$\begin{array}{ccc}
& (\underset{i}{E \circ I}) \underset{k}{F} \xrightarrow{\underset{ik}{c}} & ((E \circ I) \circ F)_n \\
\nearrow^{\rho^{-1}\ \underset{k}{F}} & \quad = \quad \nearrow^{\rho^{-1} \circ F} & \uparrow \\
\underset{i}{E} \underset{k}{F} \xrightarrow{\underset{ik}{c}} & \underset{ik}{E \circ F} & \downarrow^{\alpha} \\
\searrow^{\underset{i}{E} \lambda^{-1}} & \quad = \quad \searrow^{E \circ \lambda^{-1}} & \downarrow \\
& \underset{i}{E}(\underset{k}{I \circ F}) \xrightarrow[\underset{ik}{c}]{} & (E \circ (I \circ F))_n
\end{array}$$

and the inner commutativities indicated here are obtained from the definition of the arrow map of "∘". But the

$$\underset{ik}{c} : \underset{i}{E}\,\underset{k}{F} \to \underset{ik}{E{\circ}F}$$

for all choices with $n_{\bullet\bullet}{=}n$ form a coproduct cocone, and so the triangle in the above diagram, which is the unit coherence for $\mathrm{Dist}(\mathcal{V})$, must commute also. For E, F, G and H in $\mathrm{Dist}(\mathcal{V})$ we will now see that the corresponding associativity pentagon commutes. For each n and choice of r, p_i for all $1{\leq}i{\leq}r$, m_{ij} for all i and $1{\leq}j{\leq}p_i$, and n_{ijk} for all i, j and $1{\leq}k{\leq}m_{ij}$, such that $n_{\bullet\bullet\bullet}{=}n$, we get a diagram of the form:
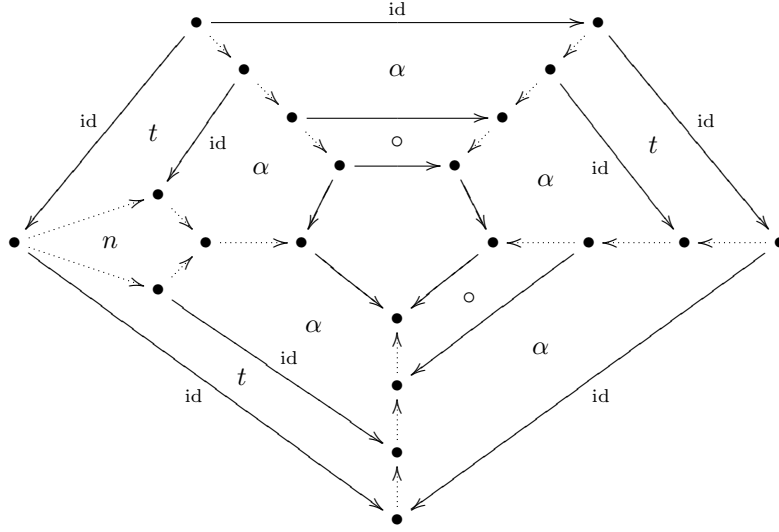
where the inner-most pentagon what we are trying to prove the commutativity of. The outer pentagon has all vertices equal to $\underset{i}{E}\,\underset{j}{F}\,\underset{k}{G}\,\underset{l}{H}$. The composites of the dotted paths of length 3, when taken over all choices, form coproduct cocones of each of the vertices of the inner pentagon. For instance for the top left vertex we have

$$\underset{i}{E}\,\underset{j}{F}\,\underset{k}{G}\,\underset{l}{H} \xdashrightarrow{cGH} (\underset{ij}{E{\circ}F})\,\underset{k}{G}\,\underset{l}{H} \xdashrightarrow{cH} ((\underset{ijk}{E{\circ}F}){\circ}G)\,\underset{l}{H} \xdashrightarrow{c} (((E{\circ}F){\circ}G){\circ}H)_n$$

and the two indicated paths involving the left most vertex are

$$\underset{i}{E}\,\underset{j}{F}\,\underset{k}{G}\,\underset{l}{H} \xdashrightarrow{cGH} (\underset{ij}{E{\circ}F})\,\underset{k}{G}\,\underset{l}{H} \xdashrightarrow{E{\circ}Fc} (\underset{ij}{E{\circ}F})(\underset{kl}{G{\circ}H}) \xdashrightarrow{c} (((E{\circ}F){\circ}G){\circ}H)_n$$

and

$$\underset{i}{E}\,\underset{j}{F}\,\underset{k}{G}\,\underset{l}{H} \xdashrightarrow{EFc} \underset{i}{E}\,\underset{j}{F}(\underset{kl}{G{\circ}H}) \xdashrightarrow{cG{\circ}H} (\underset{ij}{E{\circ}F})(\underset{kl}{G{\circ}H}) \xdashrightarrow{c} (((E{\circ}F){\circ}G){\circ}H)_n$$

and in a similar vein the reader will easily supply the details of the other dotted paths. The labels of the regions of the diagram indicate why the corresponding region commutes: "α" means the region commutes by the definition of α, "n" indicates commutativity because of naturality, "∘" indicates commutativity because of the definition of the arrow map of ∘, and "t" indicates that the region commutes trivially. The outer pentagon of course also commutes trivially. Since all this is true for all choices of the r, p_i, m_{ij} and n_{ijk}, we obtain the commutativity of the inner

pentagon since the top left dotted composites together exhibit $(((E{\circ}F){\circ}G){\circ}H)_n$ as a coproduct. The statement about $\mathrm{Mon}(\mathrm{Dist}(\mathcal{V}))$ follows immediately by unpacking the definitions involved. $\qquad\square$

4. Monads from multitensors

Multitensors generalise non-symmetric operads by example(2.6). Given certain hypotheses on the ambient braided monoidal category $\mathcal{V}$, a non-symmetric operad therein gives rise to a monad on $\mathcal{V}$ whose algebras are those of the original operad. Thus one is led to ask whether one can define a monad from a multitensor in a similar way. Such a construction is described in the present section, and we continue to assume throughout this section that $\mathcal{V}$ has coproducts.

Define the functor $\Gamma : \mathrm{Dist}(\mathcal{V}) \to \mathrm{End}(\mathcal{V})$ as

$$\Gamma(E)(X) = \coprod_{n \geq 0} \underset{1 \leq i \leq n}{\mathrm{E}}\, X$$

and so for each X in $\mathcal{V}$ we get

$$c_n : \underset{1 \leq i \leq n}{\mathrm{E}}\, X \to \Gamma(E)(X)$$

for $n \in \mathbb{N}$ forming a coproduct cocone. By the definition of I the map $c_1 : X \to \Gamma(I)(X)$ is an isomorphism, and we define that the inverses of these maps are the components of an isomorphism $\gamma_0 : 1_{\mathcal{V}} \to \Gamma(I)$. For X in $\mathcal{V}$ and m and n_i in $\mathbb{N}$ where $1 \leq i \leq m$, we can consider composites

$$\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\, X \xrightarrow{\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{c}}} \underset{i}{\mathrm{E}}\, \Gamma F X \xrightarrow{\;c_m\;} \Gamma(E)\Gamma(F)X$$

and since E is distributive all such composites exhibit $\Gamma(E)\Gamma(F)X$ as a coproduct. For X, m and n_i as above one also has composites

$$\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\, X \xrightarrow{\underset{ij}{\mathrm{c}}} (\underset{ij}{\mathrm{E}{\circ}\mathrm{F}})X \xrightarrow{\;c_{n_\bullet}\;} \Gamma(E{\circ}F)X$$

and all such composites exhibit $\Gamma(E{\circ}F)X$ as a coproduct. Thus there is a unique isomorphism γ_2 making

commute, and γ_2 is clearly natural in X.

PROPOSITION 4.1. *The data (γ_0, γ_2) make Γ into a monoidal functor, and so it sends distributive multitensors to monads. For any distributive multitensor E, one has an isomorphism $\mathrm{Mon}(E) \cong \Gamma E\text{-Alg}$ commuting with the forgetful functors into $\mathcal{V}$.*

PROOF. The definition of γ_2 in the case where $E=I$ and the $m=1$ says that the outside of

$$
\begin{array}{ccc}
\Gamma(F)X & \xrightarrow{\gamma_0\Gamma F} & \Gamma(I)\Gamma(F)X \\
\uparrow\scriptstyle{c_j} & & \searrow\scriptstyle{\gamma_2} \\
& \xrightarrow{\Gamma\lambda^{-1}} & \Gamma(I\circ F)X \\
& = & \\
\underset{j}{F} X & \xrightarrow{\lambda^{-1}} & \underset{j}{(I\circ F)}X \quad \xrightarrow{c_n} \\
\end{array}
$$

commutes for all $m \in \mathbb{N}$, and the region labelled with "=" commutes because of the definition of the arrow maps of $\circ$. Thus the inner triangle, which is the left unit monoidal functor coherence axiom, commutes also. The definition of γ_2 in the case where $F=I$ and the n_i's are all 1 says that the outside of

$$
\begin{array}{ccc}
\underset{i}{E}\Gamma(I)X & \xrightarrow{c_m} & \Gamma(E)\Gamma(I)X \\
\uparrow\scriptstyle{\underset{i}{E}\gamma_0} & = & \nearrow\scriptstyle{\Gamma(E)\gamma_0} \quad \searrow\scriptstyle{\gamma_2} \\
& & \Gamma(E)X \xrightarrow{\Gamma\rho^{-1}} \Gamma(E\circ I)X \\
& \nearrow\scriptstyle{c_m} & = \\
\underset{i\ j}{E\ I} X & \xrightarrow{\rho^{-1}} & \underset{ij}{(E\circ I)}X \quad \xrightarrow{c_{n\bullet}} \\
\end{array}
$$

commutes for all $m \in \mathbb{N}$, and the regions labelled with "=" commute because of the definition of the arrow maps of $\circ$. Thus the inner triangle, which is the right unit monoidal functor coherence axiom, commutes also. So it remains to verify that for E, F and G in $\mathrm{Dist}(\mathcal{V})$, that

$$
\begin{array}{ccc}
\Gamma(E)\Gamma(F)\Gamma(G) & \xrightarrow{\gamma_2\Gamma(G)} & \Gamma(E\circ F)\Gamma(G) \\
\scriptstyle{\Gamma(E)\gamma_2}\downarrow & & \downarrow\scriptstyle{\gamma_2} \\
\Gamma(E)\Gamma(F\circ G) & & \Gamma((E\circ F)\circ G) \\
\scriptstyle{\gamma_2}\searrow & & \swarrow\scriptstyle{\Gamma\alpha} \\
& \Gamma(E\circ(F\circ G)) & \\
\end{array}
$$

(2)

commutes. Now given X in $\mathcal{V}$ and r, m_i and n_{ij} in $\mathbb{N}$ where $1 \leq i \leq r$ and $1 \leq j \leq m_i$, one obtains a diagram of the form
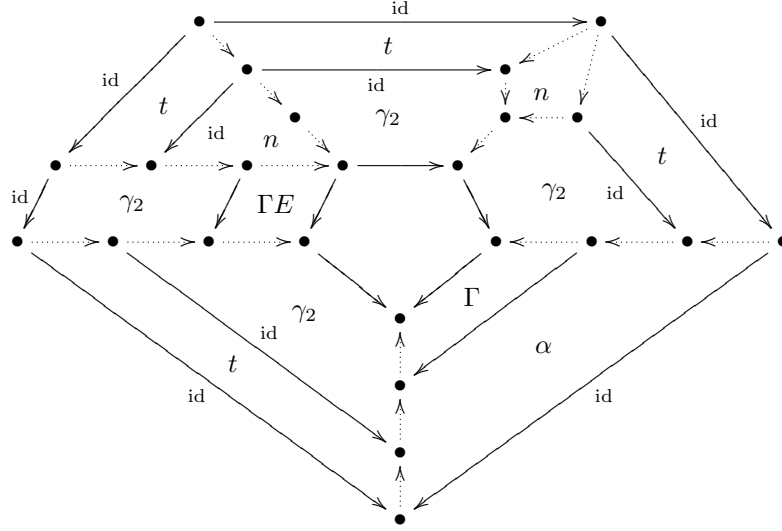
where the inner-most pentagon is (2) instantiated at X, and all the outer vertices are $\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\underset{k}{\mathrm{G}}\,X$. The two 3-fold paths into $\Gamma(E)\Gamma(F)\Gamma(G)(X)$ are the top-leftmost path

$$\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\underset{k}{\mathrm{G}}\,X \xrightarrow{\underset{i\,j\,k}{\mathrm{E\,F\,c}}} \underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\Gamma(G)X \xrightarrow{\underset{i}{\mathrm{c}}} \Gamma(E)\,\underset{j}{\mathrm{F}}\,\Gamma(G)X \xrightarrow{\Gamma(E)\,\mathrm{c}_j} \Gamma(E)\Gamma(F)\Gamma(G)(X)$$

and

$$\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\underset{k}{\mathrm{G}}\,X \xrightarrow{\underset{i\,j\,k}{\mathrm{E\,F\,c}}} \underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\Gamma(G)X \xrightarrow{\underset{i\,j}{\mathrm{E\,c}}} \underset{i}{\mathrm{E}}\,\Gamma(F)\Gamma(G)(X) \xrightarrow{\underset{i}{\mathrm{c}}} \Gamma(E)\Gamma(F)\Gamma(G)(X)$$

and these are equal because of naturality. The composites so formed by taking all choices of r, m_i and n_{ij} exhibit $\Gamma(E)\Gamma(F)\Gamma(G)(X)$ as a coproduct because E and F are distributive. The left-most dotted path into $\Gamma(E{\circ}F)\Gamma(G)(X)$ is

$$\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\underset{k}{\mathrm{G}}\,X \xrightarrow{\underset{i\,j\,k}{\mathrm{E\,F\,c}}} \underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\Gamma(G)X \xrightarrow{\underset{ij}{\mathrm{c}}} (\underset{ij}{\mathrm{E{\circ}F}})\Gamma(G)X \xrightarrow{\underset{ij}{\mathrm{c}}} \Gamma(E{\circ}F)\Gamma(G)(X)\ ,$$

the other path into $\Gamma(E{\circ}F)\Gamma(G)(X)$ is

$$\underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{F}}\,\underset{k}{\mathrm{G}}\,X \xrightarrow{\underset{ij}{\mathrm{c}}} \underset{ij}{\mathrm{E{\circ}F}}\,\underset{k}{\mathrm{G}}\,X \xrightarrow{\underset{ij}{\mathrm{E{\circ}F}}\,\underset{k}{\mathrm{c}}} (\underset{ij}{\mathrm{E{\circ}F}})\Gamma(G)X \xrightarrow{\underset{ij}{\mathrm{c}}} \Gamma(E{\circ}F)\Gamma(G)(X)\ ,$$

and similarly the reader will easily supply the definitions of the other dotted paths in the above diagram. The labelled regions of that diagram commute for the reasons indicated by the labels as with the proof of proposition(3.3), the region labelled by "Γ" commutes by the definition of the arrow map of Γ, and the region labelled by "ΓE" commutes by the definition of the arrow map of ΓE. The outer diagram commutes trivially and since this is all true for all choices of the r, m_i and n_{ij},

the inner pentagon commutes as required. The statement about $\mathrm{Mon}(E)$ follows immediately by unpacking the definitions involved.   □

EXAMPLE 4.2. One can apply proposition(4.1) to the case of example(2.6) when $(\mathcal{V}, \otimes, I)$ is a distributive braided monoidal category, because then the multitensor on $\mathcal{V}$ determined by a non-symmetric operad will also be distributive. In this way one obtains the usual construction of the monad induced by a non-symmetric operad.

EXAMPLE 4.3. Applying proposition(4.1) to the case of a distributive monoidal category $(\mathcal{V}, \otimes, I)$ as in example(3.2), one recovers the usual monoid monad $M := \Gamma(\otimes)$. In the case where $\otimes$ is cartesian product and T preserves coproducts, in view of $\Gamma(T^{\times}) = MT$ one obtains a monad structure on MT, and thus a monad distributive law $\lambda : TM \to MT$, and the algebras of MT are monoids in T-Alg by proposition(2.8). In terms of Γ and $T^{\times}$ one can describe λ explicitly. The substitution for $T^{\times}$, described in example(2.3), is a map $\mu^{\times} : T^{\times} \circ T^{\times} \to T^{\times}$ in $\mathrm{Dist}(\mathcal{V})$, and λ is the composite

$$TM \xrightarrow{\eta TM\eta} MTMT \xrightarrow{\Gamma\mu^{\times}} MT$$

in $\mathrm{End}(\mathcal{V})$.

5. Multitensors as operads

Given a cartesian monad T on a finitely complete category $\mathcal{V}$ one has the well-known notion of T-*operad* as described for example in [**12**]. There is an analogous notion of T-multitensor and we shall describe this in the present section. Under certain conditions the given monad T distributes with the monoid monad M on $\mathcal{V}$ and the composite monad MT is again cartesian, in which case one has an equivalence of categories between T-multitensors and MT-operads. The theory described in this section requires that T is a little more than cartesian, namely that it is a local right adjoint monad, and that $\mathcal{V}$ is lextensive. Both notions will be recalled here for the readers' convenience.

We recall some aspects of the theory of local right adjoints from [**15**]. In *loc.sit* such functors were called "parametric right adjoints" (p.r.a's), but local right adjoint is a better name – the prefix "local" is commonly used to denote notions that apply only on the slices of the categories involved as is the case here. A functor $T : \mathcal{A} \to \mathcal{B}$ is a *local right adjoint* (l.r.a) when for all $A \in \mathcal{A}$, the induced functors

$$T_A : \mathcal{A}/A \to \mathcal{B}/TA$$

given by applying T to arrows are right adjoints, and when $\mathcal{A}$ has a terminal object 1, this is equivalent to asking that T_1 is a right adjoint. Right adjoints are clearly l.r.a and l.r.a functors are closed under composition. Moreover one has the following simple observation which we shall use often in this work.

LEMMA 5.1. *Let I be a set and $F_i : \mathcal{A}_i \to \mathcal{B}_i$ for $i \in I$ be a family of l.r.a functors. Then*

$$\prod_i \mathcal{A}_i \xrightarrow{\prod_i F_i} \prod_i \mathcal{B}_i$$

is l.r.a.

PROOF. Given $X_i \in \mathcal{A}_i$ for $i \in I$, we have $(\prod_i F_i)_{(X_i)} = \prod_i ((F_i)_{X_i})$, which as a product of right adjoints is a right adjoint. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

There is a more explicit characterisation of l.r.a functors which is sometimes useful. A map $f : B \to TA$ is T-*generic* when for any α, β, and γ making the outside of

$$
\begin{array}{ccc}
B & \xrightarrow{\ \alpha\ } & TX \\
{\scriptstyle f}\big\downarrow & {\scriptstyle T\delta}\ \ {}^{\nearrow} & \big\downarrow{\scriptstyle T\gamma} \\
TA & \xrightarrow[\ T\beta\ ]{} & TZ
\end{array}
$$

commute, there is a unique δ for which $\gamma \circ \delta = \beta$ and $T(\delta) \circ f = \alpha$. The alternative characterisation says that T is l.r.a iff every map $f : B \to TA$ factors as

$$
B \xrightarrow{\ g\ } TC \xrightarrow{\ Th\ } TA
$$

where g is generic, and such generic factorisations are unique up to isomorphism if they exist (see [15] for more details). One defines a monad (T, η, μ) on a category $\mathcal{V}$ to be l.r.a when T is l.r.a as a functor, and η and μ are cartesian transformations. One has the following corresponding definition for multitensors.

DEFINITION 5.2. A multitensor (E, u, σ) on $\mathcal{V}$ is l.r.a when $E : \mathcal{M}\mathcal{V} \to \mathcal{V}$ is l.r.a and u and σ are cartesian transformations.

It is straight-forward to observe that E is l.r.a iff $E_n : \mathcal{V}^n \to \mathcal{V}$ is l.r.a for each $n \in \mathbb{N}$.

EXAMPLE 5.3. Let (T, η, μ) be a l.r.a monad on $\mathcal{V}$ a category with finite products. First note that $T_n^\times$ is the composite

$$
\mathcal{V}^n \xrightarrow{\ T^n\ } \mathcal{V}^n \xrightarrow{\ \Pi\ } \mathcal{V}
$$

and so is l.r.a. by lemma(5.1) and the composability of l.r.a's. From [15] lemma(2.14) the canonical maps

$$
k_{X_i} : T \prod_i X_i \to \prod_i TX_i
$$

which measure the extent to which T preserves products are cartesian natural in the X_i. Thus $T^\times$ is a l.r.a multitensor.

For a l.r.a monad (T, η, μ) on a category $\mathcal{V}$ recall that a T-*operad* is cartesian monad morphism $\alpha : A \to T$. That is, A is a monad on $\mathcal{V}$, α is a natural transformation $A \to T$ which is compatible with the monad structures, and the naturality squares of α are pullbacks. The cartesianness of α and l.r.a'ness of T implies that A is itself a l.r.a monad. For instance when $T = \mathcal{T}$ the monad on the category $\widehat{\mathbb{G}}$ of globular sets whose algebras are strict ω-categories, to be recalled in detail in section(6), T-operads are the ω-operads [1]. By analogy one has the following definition for multitensors.

DEFINITION 5.4. Let (T, η, μ) be a l.r.a monad on $\mathcal{V}$ a category with finite products. A T-*multitensor* is a cartesian multitensor morphism $\varepsilon : E \to T^\times$.

EXAMPLE 5.5. We will now unpack this notion in the case where $\mathcal{V} = \mathrm{Set}$ and T is the identity monad. Because of the pullback squares

$$
\begin{array}{ccc}
\underset{i}{\mathrm{E}}\, X_i & \xrightarrow{\;\varepsilon_{X_i}\;} & \prod_i T X_i \\
{\scriptstyle \underset{i}{\mathrm{E}}\, t_{X_i}}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \prod_i T t_{X_i}} \\
E_n 1 & \xrightarrow[\;\varepsilon_1\;]{} & (T1)^n
\end{array}
$$

the data for ε amounts to a sequence of objects $\overline{E}_n := E_n 1 \in \mathcal{V}$ for $n \in \mathbb{N}$, together with maps $\varepsilon_{n,i} : \overline{E}_n \to T1$ for $1 \le i \le n$. In this case $T1=1$ so ε amounts to a sequence $(\overline{E}_n : n \in \mathbb{N})$ of sets. In terms of this data one has

$$
(3) \qquad\qquad \underset{1\le i\le n}{\mathrm{E}}\, X_i = \overline{E}_n \times \prod_i X_i
$$

The unit of the multitensor amounts to an element $\overline{u} : 1 \to \overline{E}_1$, and the substitution σ amounts to functions

$$
\overline{\sigma}_{n_1,\dots,n_k} : \overline{E}_k \times \overline{E}_{n_1} \times \dots \times \overline{E}_{n_k} \to \overline{E}_{n_\bullet}
$$

for each finite sequence $(n_1, \dots, n_k)$ of natural numbers. The multitensor axioms for (E, u, σ) correspond to axioms that make $(\overline{E}, \overline{u}, \overline{\sigma})$ a non-symmetric operad in Set.

We assume throughout this section that $\mathcal{V}$ is lextensive. Let us now recall this notion. A category $\mathcal{V}$ is *lextensive*[3] [**5**] when it has finite limits, coproducts and for each family of objects $(X_i : i \in I)$ of $\mathcal{V}$ the functor

$$
\prod_{i \in I} \mathcal{V}/X_i \to \mathcal{V}/\left(\coprod_{i \in I} X_i \right)
$$

which sends a family of maps $(h_i : Z_i \to X_i)$ to their coproduct is an equivalence. This last property is equivalent to saying that $\mathcal{V}$ has a strict initial object and that coproducts in $\mathcal{V}$ are disjoint and stable. There are many examples of lextensive categories: for instance every Grothendieck topos is lextensive, as is CAT. Moreover if T is a coproduct preserving monad on a lextensive category $\mathcal{V}$ then T-Alg is also lextensive: for such a T the forgetful functor $T\text{-Alg} \to \mathcal{V}$ creates finite limits and coproducts, and so these exist in T-Alg and interact as nicely as they did in $\mathcal{V}$. Thus in particular the category of algebras of any higher operad is lextensive. Note in particular that lextensivity implies distributivity (see [**5**]) and so the results of the previous two sections apply in this one. The next result summarises how lextensivity interacts well with l.r.a'ness.

LEMMA 5.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be lextensive and I be a set.*

(1) *The functor $\coprod : \mathcal{A}^I \to \mathcal{A}$, which takes an I-indexed family of objects of $\mathcal{A}$ to its coproduct, is l.r.a.*

(2) *If $F_i : \mathcal{A} \to \mathcal{B}$ for $i \in I$ are l.r.a functors, then $\coprod_i F_i : \mathcal{A} \to \mathcal{B}$ is l.r.a.*

(3) *If $F_i : \mathcal{A} \to \mathcal{B}$ for $i \in I$ are functors and $\phi_i : F_i \to G_i$ are cartesian transformations, then $\coprod_i \phi_i : \coprod_i F_i \to \coprod_i G_i$ is cartesian.*

---

[3]Usually lextensivity is defined using only finite coproducts whereas we work with small ones.

PROOF. (1): given a family $(X_i : i \in I)$ of objects of $\mathcal{A}$, the functor $(\coprod)_{(X_i)}$ is just the functor

$$\prod_{i \in I} \mathcal{A}/X_i \to \mathcal{A}/\left(\coprod_{i \in I} X_i\right)$$

which is an equivalence, and thus a right adjoint.

(2): $\coprod_i F_i$ is the composite

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A}^I \xrightarrow{\prod_i F_i} \mathcal{B}^I \xrightarrow{\coprod} \mathcal{B}$$

of a right adjoint (since $\mathcal{A}$ has coproducts) followed by a l.r.a (by lemma(5.1) followed by another l.r.a (by (1), and so is l.r.a.

(3): the naturality square for $\coprod_i \phi_i$ corresponding to $f : X \to Y$ in $\mathcal{A}$ is the coproduct of the cartesian naturality squares

$$\begin{array}{ccc}
F_i X & \xrightarrow{\phi_{i,X}} & G_i X \\
{\scriptstyle F_i f}\downarrow & & \downarrow{\scriptstyle G_i f} \\
F_i Y & \xrightarrow{\phi_{i,Y}} & G_i Y
\end{array}$$

and so by (1) is itself a pullback. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Denote by $\mathrm{LraDist}(\mathcal{V})$ and $\mathrm{LraEnd}(\mathcal{V})$ the subcategories of $\mathrm{Dist}(\mathcal{V})$ and $\mathrm{End}(\mathcal{V})$ respectively, whose objects are l.r.a's and arrows are cartesian transformations.

PROPOSITION 5.7. *Let $\mathcal{V}$ be lextensive. The monoidal structure of $\mathrm{Dist}(\mathcal{V})$ restricts to $\mathrm{LraDist}(\mathcal{V})$, and Γ restricts to a strong monoidal functor*

$$\mathrm{LraDist}(\mathcal{V}) \to \mathrm{LraEnd}(\mathcal{V})$$

(which we shall also denote by Γ).

PROOF. Any functor $1 \to \mathcal{A}$ out of the terminal category is l.r.a, and thus one readily verifies that the functors $\mathcal{V}^n \to \mathcal{V}$ constant at the initial object 0 of $\mathcal{V}$ are l.r.a also. Since $1_{\mathcal{V}}$ is l.r.a the unit of $\mathrm{Dist}(\mathcal{V})$ is l.r.a. For l.r.a E and $F \in \mathrm{Dist}(\mathcal{V})$ we must verify that $E \circ F$ is l.r.a. By the formula

$$(E \circ F)_n = \coprod_{n_1 + \ldots + n_k = n} E_k(F_{n_1}, \ldots, F_{n_k})$$

and lemma(5.6) it suffices to show that each summand is l.r.a. But $E_k(F_{n_1}, \ldots, F_{n_k})$ is the composite

$$\prod_i \mathcal{V}^{n_i} \xrightarrow{\prod_i F_{n_i}} \mathcal{V}^k \xrightarrow{E_k} \mathcal{V}$$

which is l.r.a by lemma(5.1). Given $\varepsilon : E \to E'$ and $\phi : F \to F'$ in $\mathrm{LraDist}(\mathcal{V})$ we must show that $\varepsilon \circ \phi$ is cartesian. By lemma(5.6) it suffices to show that

$$E_k(F_{n_1}, \ldots, F_{n_k}) \xrightarrow{\varepsilon_k(\phi_{n_1}, \ldots, \phi_{n_k})} E'_k(F'_{n_1}, \ldots, F'_{n_k})$$

is cartesian. But this natural transformation is the composite

$$
\prod_i \mathcal{V}^{n_i} \quad \Downarrow \prod_i \phi_{n_i} \quad \mathcal{V}^k \quad \Downarrow \varepsilon_k \quad \mathcal{V}
$$

and so as a horizontal composite of cartesian transformations between pullback preserving functors, is indeed cartesian. Thus the monoidal structure of $\mathrm{Dist}(\mathcal{V})$ restricts to $\mathrm{LraDist}(\mathcal{V})$, and to finish the proof we must verify that Γ preserves l.r.a objects and cartesian transformations. Let $E \in \mathrm{Dist}(\mathcal{V})$ be l.r.a. By lemma(5.6), to establish that $\Gamma(E)$ is l.r.a it suffices to show that for all $n \in \mathbb{N}$, the functor $X \mapsto E_n(X, ..., X)$ is l.r.a, but this is just the composite

$$
\mathcal{V} \xrightarrow{\Delta} \mathcal{V}^n \xrightarrow{E_n} \mathcal{V}
$$

which is l.r.a since E_n is. Let $\phi : E \to F$ in $\mathrm{Dist}(\mathcal{V})$ be cartesian and let us see that $\Gamma(\phi)$ is cartesian. By lemma(5.6) this comes down to the cartesian naturality in X of the maps

$$
\phi_{n,X,...,X} : E_n(X, ..., X) \to F_n(X, ..., X)
$$

which is an instance of the cartesianness of ϕ_n. $\qquad\square$

EXAMPLE 5.8. From examples(5.3) and example(3.2) $T^\times$ is a l.r.a distributive multitensor when T is a coproduct preserving l.r.a monad on a lextensive category $\mathcal{V}$. By proposition(5.7), the monad MT described in example(4.3) is l.r.a and the distributive law $\lambda : TM \to MT$ is cartesian.

Modulo one last digression we are now ready to exhibit the equivalence between T-multitensors and MT-operads as promised at the beginning of this section. Recall that if $\mathcal{W}$ is a monoidal category and (M, i, m) a monoid therein, that the slice $\mathcal{W}/M$ gets a canonical monoidal structure. The unit is the unit $i : I \to M$ of the monoid, the tensor product of arrows $\alpha : A \to M$ and $\beta : B \to M$ is the composite

$$
A \otimes B \xrightarrow{\alpha \otimes \beta} M \otimes M \xrightarrow{m} M
$$

and the coherences are inherited from $\mathcal{W}$ so that the forgetful functor $\mathcal{W}/M \to \mathcal{W}$ is strict monoidal. To give $\alpha : A \to M$ a monoid structure in $\mathcal{W}/M$ is the same as giving A a monoid structure for which α becomes a monoid homomorphism, and this is just the object part of an isomorphism $\mathrm{Mon}(\mathcal{W}/M) \cong \mathrm{Mon}(\mathcal{W})/M$ commuting with the forgetful functors into $\mathcal{W}$. Moreover given a monoidal functor $F : \mathcal{W} \to \mathcal{W}'$, FM is canonically a monoid and one has a commutative square

$$
\begin{array}{ccc}
\mathcal{W}/M & \xrightarrow{F_M} & \mathcal{W}'/FM \\
\downarrow & & \downarrow \\
\mathcal{W} & \xrightarrow{F} & \mathcal{W}'
\end{array}
$$

of monoidal functors.

Applying these observations to $\Gamma : \mathrm{LraDist}(\mathcal{V}) \to \mathrm{LraEnd}(\mathcal{V})$ one obtains for each l.r.a distributive multitensor E, a monoidal functor

$$\Gamma_E : \mathrm{LraDist}(\mathcal{V})/E \to \mathrm{LraEnd}(\mathcal{V})/\Gamma E.$$

An object of $\mathrm{LraDist}(\mathcal{V})/E$ amounts to a functor $A : \mathcal{MV} \to \mathcal{V}$ together with a cartesian transformation $\alpha : A \to E$. Given such data the distributivity of A is a consequence of the cartesianness of α, the distributivity of E and the stability of $\mathcal{V}$'s coproducts. The l.r.a'ness of A is also a consequence, because the domain of any cartesian transformation into a l.r.a functor is again l.r.a. A morphism in $\mathrm{LraDist}(\mathcal{V})/E$ from α to $\beta : B \to E$ is just a natural transformation $\phi : A \to B$ such that $\beta\phi = \alpha$, because by the elementary properties of pullbacks ϕ is automatically cartesian. Thus a monoid in $\mathrm{LraDist}(\mathcal{V})/E$ is simply a cartesian multitensor morphism into E. Similarly a monoid in $\mathrm{LraEnd}(\mathcal{V})/\Gamma E$ is just a cartesian monad morphism into ΓE, and so by observing its effect on monoids in the case $E = T^\times$ where T is a coproduct preserving l.r.a monad on $\mathcal{V}$, one has a functor

$$\Gamma_T : T\text{-Mult} \to MT\text{-Op}$$

from the category of T-multitensors to the category of MT-operads.

THEOREM 5.9. *Let $\mathcal{V}$ be lextensive and T a coproduct preserving l.r.a monad on $\mathcal{V}$. Then the functor Γ_T just described is an equivalence of categories T-Mult $\simeq$ MT-Op.*

PROOF. By the way we have set things up it suffices to show that for any l.r.a distributive multitensor E on $\mathcal{V}$, the functor $\Gamma_E : \mathrm{LraDist}(\mathcal{V})/E \to \mathrm{LraEnd}(\mathcal{V})/\Gamma E$ is essentially surjective on objects and fully faithful. Let $\alpha : A \to \Gamma(E)$ be a cartesian transformation. Choosing pullbacks

$$
\begin{array}{ccc}
\underset{i}{\overline{\mathrm{A}}}\, X_i & \longrightarrow & A1 \\
{\scriptstyle \overline{\alpha}_{X_i}}\downarrow & \lrcorner & \downarrow {\scriptstyle c_n} \\
\underset{i}{\mathrm{E}}\, X_i \xrightarrow[\underset{i}{\mathrm{E}}\, t_{X_i}]{} E_n(1,...,1) & \xrightarrow[c_n]{} & \Gamma E(1)
\end{array}
$$

for each finite sequence $(X_i : 1 \leq i \leq n)$ of objects of $\mathcal{V}$, one obtains a cartesian transformation $\overline{\alpha} : \overline{A} \to E$. The stability of $\mathcal{V}$'s coproducts applied to the pullbacks

$$
\begin{array}{ccc}
\overline{A}_n(1,...,1) & \xrightarrow{\ \overline{\alpha}\ } & E_n(1,...,1) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle c_n} \\
A1 & \xrightarrow[\alpha_1]{} & \Gamma E(1)
\end{array}
$$

for each $X \in \mathcal{V}$ and $n \in \mathbb{N}$ ensures that $\Gamma_E(\overline{\alpha}) \cong \alpha$ thus verifying essential surjectivity. Let $\alpha : A \to E$ and $\beta : B \to E$ be cartesian, and $\phi : \Gamma A \to \Gamma B$ such that $\Gamma(\beta)\phi = \Gamma\alpha$. To finish the proof we must show there is a unique $\phi' : A \to B$ such that $\beta\phi' = \alpha$ and $\Gamma\phi' = \phi$. The equation $\Gamma\phi' = \phi$ implies in particular that $\coprod_n \phi'_{n,1} = \phi_1$, and

this determines the components $\phi'_{n,1,\dots,1}$ uniquely because of

$$
\begin{array}{ccccc}
A_n(1,\dots,1) & \xrightarrow{\phi'_{n,1,\dots,1}} & B_n(1,\dots,1) & \xrightarrow{\beta_{n,1,\dots,1}} & E_n(1,\dots,1) \\
\downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle c_n} \\
\Gamma A(1) & \xrightarrow{\phi_1} & \Gamma B(1) & \xrightarrow{\amalg \beta_n} & \Gamma E(1)
\end{array}
$$
with top arc $\alpha_{n,1,\dots,1}$ and bottom arc $\amalg\alpha_n$.

and these components determine ϕ' uniquely because of

$$
\begin{array}{ccccc}
\underset{i}{A}\,X_i & \xrightarrow{\phi'_{X_i}} & \underset{i}{B}\,X_i & \xrightarrow{\beta_{X_i}} & \underset{i}{E}\,X_i \\
\downarrow{\scriptstyle \underset{i}{A}\,t_{X_i}} & & \downarrow{\scriptstyle \underset{i}{B}\,t_{X_i}} & & \downarrow{\scriptstyle \underset{i}{E}\,tX_i} \\
\underset{i}{E}\,1 & \xrightarrow{\phi'_{n,1,\dots,1}} & \underset{i}{B}\,1 & \xrightarrow{\beta_{n,1,\dots,1}} & \underset{i}{A}\,1
\end{array}
$$
with top arc α_{X_i} and bottom arc $\alpha_{n,1,\dots,1}$.

and the equation $\beta\phi'=\alpha$. To see that $\Gamma\phi'=\phi$, that is $\coprod_n \phi'_{n,X,\dots,X}=\phi_X$ for all $X \in \mathcal{V}$, one deduces that the inner square in

$$
\begin{array}{ccccccc}
A_n(1,\dots,1) & & \xrightarrow{\phi'_{n,1,\dots,1}} & & & & B_n(1,\dots,1) \\
 & \nwarrow{\scriptstyle A_n(t_X,\dots,t_X)} & & & & \nearrow{\scriptstyle B_n(t_X,\dots,t_X)} & \\
 & & A_n(X,\dots,X) & \xrightarrow{\phi'_{n,X,\dots,X}} & B_n(X,\dots,X) & & \\
\downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle c_n} \\
 & & \Gamma A(X) & \xrightarrow{\phi_X} & \Gamma B(X) & & \\
 & \swarrow{\scriptstyle \Gamma A(t_X)} & & & & \searrow{\scriptstyle \Gamma B(t_X)} & \\
\Gamma A(1) & & & \xrightarrow{\phi_1} & & & \Gamma B(1)
\end{array}
$$

is a pullback since the outer square and all other regions in this diagram are pullbacks, and so the result follows by lextensivity. $\qquad\square$

6. The strict ω-category monad

The setting of the previous section involved a coproduct preserving l.r.a monad T, and after this section we shall be concerned with the case where $T = \mathcal{T}$ the strict ω-category monad on $\widehat{\mathbb{G}}$ the category of globular sets, and its finite dimensional analogues the strict n-category monads. We give a precise and purely inductive combinatorial description of $\mathcal{T}$ in section(6.2), using some further theory of l.r.a monads on presheaf categories which we develop in section(6.1), to facilitate our description of the details.

6.1. Specifying l.r.a monads on presheaf categories. From [**15**] we know that to specify a l.r.a $T : \widehat{\mathbb{B}} \to \widehat{\mathbb{C}}$ one can begin with $P \in \widehat{\mathbb{C}}$ and a functor $E_T :$ el$(P) \to \widehat{\mathbb{B}}$. Here we will usually not distinguish notationally between $p \in PC$ and $E_T(p, C)$. Given $k : D \to C$ in $\mathbb{C}$ we shall denote by pk the element $Pk(p)$ and by $\overline{k} : pk \to p$ the map

$$E_T(k : (pk, D) \to (p, C)).$$

Given this data one can then define an element of $TX(C)$ to be a pair (p, h) where $p \in PC$ and $h : p \to X$ in $\widehat{\mathbb{B}}$. For a map $k : D \to C$ one defines $TX(k)(p, h) = (pk, h\overline{k})$, and one identifies $P = T1$. If the $E_T(p, C)$ are all connected, then T preserves coproducts.

With T so specified it is not hard to characterise generic morphisms. To give a map $f : A \to TX$ is to give for $a \in AC$ an element $p_a \in PC$ together with a map $f_a : p_a \to X$ in $\widehat{\mathbb{B}}$, and this data should be natural in C. The assignment $(C, a) \mapsto p_a$ is the object map of a functor $\overline{f} : \text{el}(A) \to \widehat{\mathbb{B}}$ and the f_a are the components of a cocone with vertex X. Factoring this cocone through its colimit Z gives a factorisation

$$A \xrightarrow{\; g \;} TZ \xrightarrow{\; Th \;} TX$$

where the g_a are the components of the universal cocone. One can easily verify directly that such a g is generic, and since generic factorisations are unique up to isomorphism, one obtains

LEMMA 6.1. *For $T : \widehat{\mathbb{B}} \to \widehat{\mathbb{C}}$ specified as above, $f : A \to TX$ is generic iff its associated cocone exhibits X as a colimit.*

EXAMPLES 6.2.   (1) If in particular A is a representable C, then $f : A \to TX$ amounts to a pair $(p, h : p \to X)$. The associated cocone consists of the one map p and so f is generic in this case iff p is an isomorphism.
(2) In the case $T = 1_{\widehat{\mathbb{C}}}$, $f : A \to X$ is generic iff it is an isomorphism.
(3) Given $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ specified as above, a morphism $f : C \to T^2 X$ amounts to a pair $(p, h : p \to TX)$. This morphism is T^2-generic iff h is T-generic because to give a commuting diagram as depicted on the left

$$
\begin{array}{ccc}
C & \xrightarrow{\;\alpha\;} & T^2 Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T^2\gamma} \\
T^2 X & \xrightarrow[T^2\beta]{} & T^2 Z
\end{array}
\qquad\qquad
\begin{array}{ccc}
p & \xrightarrow{\;\alpha'\;} & T Y \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle T\gamma} \\
T X & \xrightarrow[T^2\beta]{} & T Z
\end{array}
$$

is the same as giving a commuting diagram as depicted on the right in the previous display, and so the assertion follows by definition of "generic".

Suppose now that such a $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ comes with a cartesian transformation $\eta : 1 \to T$. The component η_1 picks out elements $u_C \in PC$ and for all $X \in \widehat{\mathbb{C}}$ the naturality of η with respect to the map $X \to 1$ shows that the components of η have the explicit form

$$x \in XC \mapsto (u_C, x' : u_C \to X).$$

Observing

$$
\begin{array}{ccc}
u_C C \xrightarrow{\ \eta\ } Tu_C(C) & \qquad & \iota \dashrightarrow (u_C, 1_{u_C}) \\
{\scriptstyle x'}\downarrow \qquad \downarrow {\scriptstyle Tx'} & & \qquad\qquad\ \downarrow \\
XC \xrightarrow[\ \eta\ ]{} TX(C) & & x \longmapsto (u_C, x')
\end{array}
$$

we have a unique element of $\iota \in u_C C$ which is sent by η to 1_{u_C}. It is a general fact [**14**] that components of cartesian transformations reflect generic morphisms, and so by examples(6.2)(1) and (2) the morphism $C{\to}u_C$ corresponding to ι is an isomorphism. One may assume that this isomorphism is an identity by redefining the yoneda embedding if necessary to agree with $C \mapsto u_C$ and similarly on arrows, so we shall write $C = u_C$. Then the components of η may be written as

$$ x \mapsto (C, x : C{\to}X) $$

where the x on the right hand side corresponds to the x on the left hand side by the yoneda lemma.

DEFINITION 6.3. Let T be a l.r.a endofunctor of $\widehat{\mathbb{C}}$ and $\eta : 1{\to}T$ be a cartesian transformation. A pair (P, E_T) giving the explicit description of (T, η) as above is called a *specification* of (T, η). That is to say, a specification consists of $P \in \widehat{\mathbb{C}}$, $E_T : \mathrm{el}(P){\to}\widehat{\mathbb{C}}$ and an inclusion $\mathbb{C}{\hookrightarrow}\mathrm{el}(P)$ whose composite with E_T is the yoneda embedding.

By the discussion preceeding definition(6.3) every such (T, η) has a specification. Let us denote the assignments of an arbitary natural transformation $\mu : T^2{\to}T$ by

$$ (p \in PC, f : p{\to}TX) \mapsto (q_f \in PC, h_f : q_f{\to}X). $$

Naturality of μ in C says that for $k : D{\to}C$, $q_{f\overline{k}} = q_f k$ and $h_{f\overline{k}} = h_f \overline{k}$. Naturality of μ in X says that for $h : X{\to}Y$, $q_{T(h)f} = q_f$ and $h_{T(h)f} = hh_f$. Suppose that μ is cartesian. Observing

$$
\begin{array}{ccc}
T^2 q_f \xrightarrow{\ \mu\ } Tq_f & \qquad & g_f \dashrightarrow (q_f, 1_{q_f}) \\
{\scriptstyle T^2 h_f}\downarrow \qquad \downarrow {\scriptstyle Th_f} & & \qquad\qquad\ \downarrow \\
T^2 X \xrightarrow[\ \mu\ ]{} TX & & (p, f : p{\to}TX) \longmapsto (q_f, h_f)
\end{array}
$$

one finds that $\forall p \in PC$ and $f : p{\to}TX$, $\exists! g_f : p{\to}Tq_f$ such that $f = T(h_f)g_f$ and $h_{g_f} = $ id. By example(6.2)(3) and the fact that cartesian transformations reflect generics, such g_f's are automatically generic. Conversely given such g_f's one can readily verify that the naturality squares of μ corresponding to maps $X{\to}1$ are pullbacks and so verify that μ is cartesian. We record these observations in

LEMMA 6.4. *Let (T, η) be specified as in definition(6.3). To give a cartesian natural transformation $\mu : T^2{\to}T$ is to give for each $p \in PC$ and $f : p{\to}TX$, an element $q_f \in PC$ and a factorisation*

$$ p \xrightarrow{\ g_f\ } Tq_f \xrightarrow{\ Th_f\ } TX $$

satisfying

    (1) *For $k : D{\to}C$, $q_{f\overline{k}} = q_f k$ and $h_{f\overline{k}} = h_f \overline{k}$.*
    (2) *For $h : X{\to}Y$, $q_{T(h)f} = q_f$ and $h_{T(h)f} = hh_f$.*

(3) *For all $p \in PC$ and $f : p \to TX$, g_f is unique such that $f = T(h_f)g_f$ and $h_{g_f} = \mathrm{id}$.*

and given this data, the g_f are automatically generic morphisms.

Thus a cartesian transformation $\mu : T^2 \to T$ amounts to a nice choice of certain generic factorisations for T. Given such a characterisation it is straight-forward to unpack what the monad axioms for (T, η, μ) say in terms of these factorisations.

LEMMA 6.5. *Let (T, η) be specified as in definition(6.3). To give $\mu : T^2 \to T$ making (T, η, μ) a l.r.a monad is to give factorisations as in lemma(6.4) which satisfy the following further conditions:*

(1) *For all $p \in PC$ and $f : p \to X$, $q_{\eta f} = p$ and $h_{\eta f} = f$.*
(2) *For all $p \in PC$ and $f : p \to X$, $q_{(p,f)} = p$ and $h_{(p,f)} = f$ where (p, f) denotes the map $C \to TX$ corresponding to the element $(p, f) \in TX(C)$ by the yoneda lemma.*
(3) *For all $p \in PC$ and $f : p \to TX$, $q_{h_f} = q_{\mu f}$ and $h_{h_f} = h_{\mu f}$.*

To summarise, given a specification of a l.r.a $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, one has for each $C \in \mathbb{C}$ and $f : C \to TX$, $p \in PC$ and a generic factorisation

$$C \xrightarrow{\; g \;} Tp \xrightarrow{\; Th \;} TX$$

of f. The data of a l.r.a monad (T, η, μ) enables us to regard $C \in PC$ and gives us for each $p \in PC$ and $f : p \to TX$, a choice of $q_f \in PC$ and generic factorisation

$$p \xrightarrow{\; g_f \;} Tq_f \xrightarrow{\; Th_f \;} TX$$

of f, and these choices satisfy certain axioms.

In the case of the strict ω-category monad below some further simplifications are possible enabling one to dispense with need to verify the additional conditions of lemma(6.5) when describing it. The reason as we shall see, is that this case conforms to the following definition.

DEFINITION 6.6. A l.r.a $T : \widehat{\mathbb{B}} \to \widehat{\mathbb{C}}$ specified by $E_T : \mathrm{el}(P) \to \widehat{\mathbb{B}}$ is *tight* when for all p and $q \in PC$ and $\iota : p \cong q$ in $\widehat{\mathbb{B}}$, one has $p = q$ in PC and $\iota = \mathrm{id}$.

Clearly tightness is a property of T, that is, is independent of the specification.

EXAMPLES 6.7.   (1) Let T be the free monoid endofunctor of Set. Then E_T is a functor $\mathbb{N} \to$ Set sending $n \in \mathbb{N}$ to a set with n elements. There are of course many non-trivial automorphisms of a finite set, and so T is not tight.
(2) Let T be the free category endofunctor on Graph which we regard as presheaves on $0 \rightrightarrows 1$. Then $P0 = \{0\}$ and $P1 = \{[n] : n \in \mathbb{N}\}$ and the graph $[n]$ has object set $\{i : 0 \leq i \leq n\}$ and a unique edge $(i-1) \to i$ for each $1 \leq i \leq n$. With these details at hand one readily verifies that this T is tight.
(3) The free symmetric multicategory endofunctor on the category of multigraphs as described in example(2.14) of [**14**] is not tight. In this case one actually has distinct p and q in PC sent by E_T to isomorphic multigraphs.

LEMMA 6.8. *If $T : \widehat{\mathbb{B}} \to \widehat{\mathbb{C}}$ is a tight l.r.a then for all $A : \widehat{\mathbb{B}} \to \widehat{\mathbb{C}}$ there exists at most one cartesian transformation $A \to T$.*

PROOF. Let α and $\beta : A{\to}T$ be cartesian transformations and $a \in AX(C)$. For a given specification P one has p_α and a generic factorisation

$$
\begin{array}{ccc}
C \xrightarrow{\ a\ } AX & \xrightarrow{\ \alpha_X\ } & TX \\
\end{array}
$$

$$
C \xrightarrow{\ a\ } AX \xrightarrow{\ \alpha_X\ } TX
$$

with g_α down-left and Th_α up-right to Tp_α

and using the cartesian naturality square for α corresponding to h_α, one has $g'_\alpha : C{\to}Ap_\alpha$ unique such that $\alpha g'_\alpha = g_\alpha$ and $a = A(h_\alpha)g'_\alpha$. Since cartesian transformations reflect generics, this last equation is an A-generic factorisation of a, and similarly one obtains another one: $a = A(h_\beta)g'_\beta$ by using β instead of α. Thus there is a unique isomorphism $\delta : p_\alpha{\to}p_\beta$ so that $A(\delta)g'_\alpha = g'_\beta$ and $h_\alpha\delta = h_\beta$. By tightness δ is an identity and so $\alpha_X a = \beta_X a$. $\qquad\square$

Thus given a tight l.r.a $T : \widehat{\mathbb{C}}{\to}\widehat{\mathbb{C}}$, cartesian transformations $\eta : 1{\to}T$ and $\mu : T^2{\to}T$ are unique if they exist, and when they do the monad axioms for (T, η, μ) are automatic. This gives the following refinement of lemma(6.5) in the tight case.

COROLLARY 6.9. *Let (T, η) be specified as in definition(6.3) and let T be tight. To give $\mu : T^2{\to}T$ making (T, η, μ) a l.r.a monad is to give factorisations as in lemma(6.4).*

Moreover for a tight l.r.a monad T on $\widehat{\mathbb{C}}$, the multitensor $T^\times$ admits the same simplifications.

LEMMA 6.10. *Let (T, η, μ) be a l.r.a monad on $\widehat{\mathbb{C}}$ such that T is tight. Then for all $E : \mathcal{M}\widehat{\mathbb{C}}{\to}\widehat{\mathbb{C}}$, there exists at most one cartesian transformation $\varepsilon : E{\to}T^\times$.*

PROOF. To give such an ε is to give for each $n \in \mathbb{N}$ a cartesian transformation $\varepsilon_n : E_n{\to}T_n^\times$, and so it suffices by lemma(6.8), to show that $T_n^\times : \widehat{\mathbb{C}}^n{\to}\widehat{\mathbb{C}}$ is tight for all $n \in \mathbb{N}$. The functor $E_{T_n^\times}$ has object map $((p_1, ..., p_n), C) \mapsto (p_1, ..., p_n)$. For $q_1, ..., q_n \in T1(C)$, to give an isomorphism $\iota : (p_1, ..., p_n) \cong (q_1, ..., q_n)$ in $\widehat{\mathbb{C}}^n$, is to give isomorphisms $\iota_i : p_i \cong q_i$ for $1{\leq}i{\leq}n$, in which case the ι_i are identities by the tightness of T, and so $T_n^\times$ is also tight. $\qquad\square$

Thus for a tight monad T on $\widehat{\mathbb{C}}$, being a T-operad is actually a *property* of a monad on $\widehat{\mathbb{C}}$, and similarly for T-multitensors. We shall exploit this observation notationally below, for instance, by denoting a T-operad $\alpha : A{\to}T$ as we just have as a monad morphism, or just by referring to the monad A, depending on what is most convenient for the given situation.

6.2. Inductive description of the strict ω-category monad. A goal of this paper to clarify the inductive nature of the operadic approach to higher category theory of [1]. The starting point of that approach is a precise description of the monad $(\mathcal{T}, \eta, \mu)$ on the category $\widehat{\mathbb{G}}$ of globular sets whose algebras are strict ω-categories. Thus in this section we recall this monad, but describe it a little differently to the way it has been described in the past. We shall give here a purely inductive description of this fundamental object, and we shall use the results of the previous section to expedite our account of the details. That the algebras for the monad described in this section really are strict ω-categories defined in the usual

way by successive enrichments, is presented in section(8) as a pleasant application of our general theory.

The category $\mathbb{G}$ has as objects natural numbers and for $n < m$ maps

$$n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} m$$

and these satisfy $\sigma\tau = \tau\tau$ and $\tau\sigma = \sigma\sigma$. Thus an object of the category $\widehat{\mathbb{G}}$ of globular sets is a diagram

$$X_0 \overset{s}{\underset{t}{\leftleftarrows}} X_1 \overset{s}{\underset{t}{\leftleftarrows}} X_2 \overset{s}{\underset{t}{\leftleftarrows}} X_3 \overset{s}{\underset{t}{\leftleftarrows}} \cdots$$

of sets and functions such that $ss = st$ and $ts = tt$. The elements of X_n are called n-cells, and for an $(n+1)$-cell x, the n-cells sx and tx are called the source and target of x respectively. In fact for each $k \leq n$, we can define source and target k-cells of x and we denote these by $s_k x$ and $t_k x$, only dropping the indexing when there is little risk of confusion. Given a pair (a, b) of n-cells of X, one can define the globular set $X(a, b)$. A k-cell of $X(a, b)$ is an $(n + k)$-cell x of X such that $s_k x = a$ and $t_k x = b$. Sources and targets for $X(a, b)$ are inherited from X. In particular the globular sets $X(a, b)$ where a and b are 0-cells are called *the homs of* X. A morphism $f : X \to Z$ of globular sets induces maps $X(a, b) \to Z(f_0 a, f_0 b)$ on the homs. Conversely, to give f it suffices to specify a function $f_0 : X_0 \to Z_0$ and for all $a, b \in X_0$, morphisms $X(a, b) \to Z(f_0 a, f_0 b)$ of globular sets.

A finite sequence $(X_1, ..., X_n)$ of globular sets may be regarded as a globular set, whose set of 0-cells is $\{i \in \mathbb{N} : 0 \leq i \leq n\}$ and whose only non-empty homs are given by $(X_1, ..., X_n)(i - 1, i) = X_i$ for $1 \leq i \leq n$. This construction is the object map of a functor $\widehat{\mathbb{G}}^n \to \widehat{\mathbb{G}}$.

We now begin our description of the endofunctor $\mathcal{T}$ in the spirit of section(6.1). The role of P is played by the globular set Tr of trees. The set Tr_0 contains one element denoted as 0 and its associated globular set contains one 0-cell, also called 0, and nothing else. By induction an element of Tr_{n+1} is a finite sequence $(p_1, ..., p_k)$ of elements of Tr_n and its associated globular set is just the sequence of globular sets $(p_1, ..., p_k)$ regarded as a globular set as in the previous paragraph. So far we have defined the elements of Tr_n for all n and the object map of $E_{\mathcal{T}} : \mathrm{el}(\mathrm{Tr}) \to \widehat{\mathbb{G}}$. We denote by $\sigma : 0 \to p$ the map which selects the object $0 \in p$, and by $\tau : 0 \to p$ the map which selects the maximum vertex of p (using $\leq$ inherited from $\mathbb{N}$).

The source and target maps $s, t : \mathrm{Tr}_{n+1} \to \mathrm{Tr}_n$ coincide and are denoted as ∂. For each n we must define this map and give maps $\sigma : \partial p \to p$ and $\tau : \partial p \to p$ which satisfy the equations $\sigma\sigma = \tau\sigma$ and $\tau\tau = \sigma\tau$ in

(4)
$$\partial^2 p \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \partial p \underset{\tau}{\overset{\sigma}{\rightrightarrows}} p$$

for all $p \in \mathrm{Tr}_{n+2}$, in order to complete the description of Tr and the functor $E_{\mathcal{T}}$, and thus the definition of $\mathcal{T}$. The maps ∂, σ and τ are given by induction as follows. For the initial step ∂ is uniquely determined since Tr_0 is singleton and σ and τ are as described in the previous paragraph. For the inductive step let $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+2}$. Then $\partial p = (\partial p_1, ..., \partial p_k)$ and the maps $\sigma, \tau : \partial p \to p$ are the identities on 0-cells, and the non-empty hom maps are given by $\sigma, \tau : \partial p_i \to p_i$ respectively for $1 \leq i \leq k$. The verification of $\sigma\sigma = \tau\sigma$ and $\tau\tau = \sigma\tau$ as in (4) is given

by induction as follows. The initial step when $n = 0$ is clear since $\partial^2 p = 0$, and the 0-cell maps of $\sigma, \tau : \partial p \to p$ are both the identity. For the inductive step let $p \in \mathrm{Tr}_{n+3}$, then all the maps in (4) are identities on 0-cells, and on the homs the desired equations follow by induction.

By section(6.1) we have completed the description of a l.r.a $\mathcal{T} : \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ and we will now see that it is tight. Once again we argue by induction on n. In the case $n = 0$ the result follows because $\mathrm{Tr}_0 = \{0\}$ and the only automorphism of $0 \in \widehat{\mathbb{G}}$ is the identity. For the inductive step let $p, q \in \mathrm{Tr}_{n+1}$ and suppose that one has $\iota : p \cong q$ in $\widehat{\mathbb{G}}$. Since the only non-empty homs for p and q are between consecutive elements of their vertex sets, any $f : p \to q$ in $\widehat{\mathbb{G}}$ is order preserving in dimension 0. Thus the 0-cell map of ι is an order preserving bijection, and so must be the identity. The hom maps of ι must also be identities by induction. Since the globular sets associated to $p \in \mathrm{Tr}_n$ are also connected we have the following result.

PROPOSITION 6.11. $\mathcal{T} : \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ defined as follows is l.r.a, tight and coproduct preserving:

- an n-cell of $\mathcal{T}X$ is a pair $(p, f : p \to X)$ where $p \in \mathrm{Tr}_n$.
- for $n \geq 1$, $s(p, f) = (\partial p, f\sigma)$ and $t(p, f) = (\partial p, f\tau)$.
- for $h : X \to Y$, $\mathcal{T}(h)(p, f) = (p, hf)$.

We will now specify the cartesian unit $\eta : 1 \to \mathcal{T}$, and from section(6.1) we know that this amounts to factoring the yoneda embedding through $E_{\mathcal{T}}$. We already have $0 \in \mathrm{Tr}_0$, and by induction we define $n + 1 = (n) \in \widehat{\mathbb{G}}$. Notice that the set of k-cells of n is $\{0, 1\}$ when $k < n$ and $\{0\}$ when $k = n$. Moreover by an easy inductive proof the reader may verify that the k-cell maps of $\sigma : n \to n + 1$ and $\tau : n \to n + 1$ are the identities for $k < n$, and pick out 0 and 1 respectively when $k = n$. One has functions $\mathrm{ev}_0 : \widehat{\mathbb{G}}(n, X) \to X_n$ given by $f \mapsto f_n(0)$ clearly natural in $X \in \widehat{\mathbb{G}}$. By another easy induction one may verify that these functions are bijective, and natural in n in the sense that $sf_{n+1}(0) = (f\sigma)_n(0)$ and $tf_{n+1}(0) = (f\tau)_n(0)$. Henceforth we regard the identification of n as a globular set in this way as *the* yoneda embedding, and the components of η are given by $x \in X_n \mapsto x : n \to X$.

Before specifying the multiplication $\mu : \mathcal{T}^2 \to \mathcal{T}$ some preliminary remarks are in order. For 0-cells a and b of X, an n-cell of the hom $\mathcal{T}X(a, b)$ consists by definition, of $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$ together with $f : p \to X$ such that $f\sigma = a$ and $f\tau = b$. In other words one has a sequence $(x_0, ..., x_k)$ of 0-cells of X such that $x_0 = a$ and $x_k = b$, together with maps $f_i : p_i \to X(x_{i-1}, x_i)$ for $1 \leq i \leq k$. Another way to say all this is that for a given sequence $(x_0, ..., x_k)$ of 0-cells of X such that $x_0 = a$ and $x_k = b$, one has an inclusion

$$c_{x_i} : \prod_{1 \leq i \leq k} \mathcal{T}(X(x_{i-1}, x_i)) \to \mathcal{T}X(a, b)$$

in $\widehat{\mathbb{G}}$, and the following result.

LEMMA 6.12. *The maps c_{x_i}, for all sequences $(x_0, ..., x_k)$ of 0-cells of X such that $x_0 = a$ and $x_k = b$, form a coproduct cocone.*

Let $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$. A map $f : p \to \mathcal{T}X$ amounts to 0-cells fi of X for $0 \leq i \leq k$, together with hom maps $f_i : p_i \to \mathcal{T}X(f(i-1), fi)$ for $1 \leq i \leq k$. Since the p_i are connected, the f_i amount to 0-cells $(x_{i0}, ..., x_{im_i})$ of X such that $x_{i0} = f(i-1)$ and $x_{im_i} = fi$, together with maps $f_{ij} : p_i \to \mathcal{T}(X(x_{(ij)-1}, x_{ij}))$ for $1 \leq i \leq k$ and

$1 \leq j \leq m_i$ where

$$(i,j) - 1 = \begin{cases} (i, j-1) & \text{when } j > 0. \\ (i-1, m_{i-1}) & \text{when } j = 0 \text{ and } i > 0. \\ 0 & \text{when } i = j = 0. \end{cases}$$

In other words for $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$, to give $f : p \to \mathcal{T}X$ is to give objects x_0 and x_{ij} of X together with maps $f_{ij} : p_i \to \mathcal{T}(X(x_{(ij)-1}, x_{ij}))$ for $1 \leq i \leq k$ and $1 \leq j \leq m_i$. We shall call x_0 and the x_{ij} *the 0-cells of f*, and the f_{ij} the *hom map components* of f. Observe that for $h : X \to Y$, the 0-cells of $\mathcal{T}(h)f$ are given by hx_0 and hx_{ij}, and the hom map components by hf_{ij} where $1 \leq i \leq k$ and $1 \leq j \leq m_i$.

Now we specify the multiplication $\mu : \mathcal{T}^2 \to \mathcal{T}$ following lemma(6.4). For $p \in \mathrm{Tr}_n$ and $f : p \to \mathcal{T}X$ the factorisation of f that we must provide will be given by induction on n. When $n = 0$, $p = 0$ and a map $f : 0 \to \mathcal{T}X$ picks out a 0-cell $(0, x : 0 \to X)$ of $\mathcal{T}X$. Define $q_f = 0$, $h_f = x$ and $g_f : 0 \to \mathcal{T}0$ to pick out $(0, 1_0)$. For the inductive step let $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$ and $f : p \to \mathcal{T}X$. Then define

$$q_f = (q_{f_{ij}} : 1 \leq i \leq k, 1 \leq j \leq m_i)$$

where the f_{ij} are the hom map components of f as defined in the previous paragraph. Define h_f to have 0-cell mapping given by $0 \mapsto x_0$ and $(i, j) \mapsto x_{ij}$, and hom maps by $h_{f_{ij}}$. Define g_f to have underlying 0-cells given by 0 and (i, j), and hom map components by $g_{f_{ij}}$. By definition we have $f = \mathcal{T}(h_f)g_f$.

PROPOSITION 6.13. $(\mathcal{T}, \eta, \mu)$ *with $\mathcal{T}$ as specified in proposition(6.11), and η and μ given by*

$$x \in X_n \mapsto (n, x : n \to X) \qquad (p \in \mathrm{Tr}_n, f : p \to \mathcal{T}X) \mapsto (q_f, h_f)$$

is a l.r.a monad.

PROOF. By corollary(6.9) it suffices to verify conditions (1)-(3) of lemma(6.4). Condition(1) says that for $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$ and $f : p \to \mathcal{T}X$: $q_{f\sigma} = q_{f\tau} = \partial q_f$, $h_{f\sigma} = h_f \sigma$ and $h_{f\tau} = h_f \tau$. Let us write x_0 and x_{ij} for the 0-cells of f and f_{ij} for the hom map components where $1 \leq i \leq k$ and $1 \leq j \leq m_i$. In the case $n = 0$, we must have $0 = q_{f\sigma} = q_{f\tau} = \partial q_f$ since 0 is the only element of Tr_0. Clearly $f\sigma$ picks out x_0 and $f\tau$ picks out x_{km_k}, and so $h_{f\sigma} : 0 \to X$ picks out x_0 and $h_{f\tau} : 0 \to X$ picks out x_{km_k} by the initial step of the description of the factorisations. By the definition of the object map of h_f, $h_f \sigma$ and $h_f \tau$ also pick out the 0-cells x_0 and x_{km_k} respectively, thus verifying the $n = 0$ case of condition(1). For the inductive step let $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+2}$ and $f : p \to \mathcal{T}X$. First note that $\sigma, \tau : \partial p \to p$ are identities on 0-cells and so f, $f\sigma$ and $f\tau$ have the same 0-cells which we are denoting by x_0 and x_{ij}. Moreover by the definition of hom map components, one has $(f\sigma)_{ij} = f_{ij}\sigma$ and $(f\tau)_{ij} = f_{ij}\tau$. Thus by induction

$$q_{f\sigma} = (q_{f_{ij}\sigma} : 1 \leq i \leq k, 1 \leq j \leq m_i) = (\partial q_{f_{ij}} : 1 \leq i \leq k, 1 \leq j \leq m_i) = \partial q_f$$

and similarly $q_{f\tau} = \partial q_f$. Since $\sigma, \tau : \partial q_f \to q_f$ are identities on 0-cells the equations $h_{f\sigma} = h_f \sigma$ and $h_{f\tau} = h_f \tau$ are true on 0-cells, and on homs these equations follow by induction.

Condition(2) says that for $p \in \mathrm{Tr}_n$, $f : p \to \mathcal{T}X$ and $h : X \to Y$, $q_{\mathcal{T}(h)f} = q_f$ and $h_{\mathcal{T}(h)f} = hh_f$. When $n = 0$ these equations are immediate. For the inductive step let $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$, $f : p \to \mathcal{T}X$ and $h : X \to Y$. The objects of $q_{\mathcal{T}(h)f}$ and q_f coincide by definition, and the homs do by induction. The object maps of $h_{\mathcal{T}(h)f}$ and hh_f coincide by definition and their homs maps coincide by induction.

Condition(3) says that for $p \in \mathrm{Tr}_n$ and $f : p {\rightarrow} \mathcal{T}X$, g_f is unique such that $f = \mathcal{T}(h_f)g_f$ and $h_{g_f} = \mathrm{id}$. For $n = 0$ this is clear by inspection. For the inductive step let $p = (p_1, ..., p_k) \in \mathrm{Tr}_{n+1}$ and $f : p {\rightarrow} \mathcal{T}X$. By inspection the 0-cell map of h_{g_f} is the identity, and by induction its hom maps are also identities. As for uniqueness, the object map of g_f is determined uniquely by k and $m_i \in \mathbb{N}$ for $1{\leq}i{\leq}k$, and the uniqueness of the hom maps follows by induction. $\qquad\square$

7. Normalised $\mathcal{T}$-operads and $\mathcal{T}$-multitensors

In this section we relate $\mathcal{T}$-operads to $\mathcal{T}$-multitensors and so express $\mathcal{T}$-operad algebras as enriched categories. Under a mild condition on an operad $\alpha : A{\rightarrow}\mathcal{T}$, that it be *normalised* in the sense to be defined shortly, one can construct a multitensor $\overline{A}$ on $\widehat{\mathbb{G}}$ such that $\overline{A}$-categories are A-algebras. Moreover $\overline{A}$ is in fact a $\mathcal{T}$-multitensor, and the construction $\overline{(\ )}$ is part of an equivalence of categories between $\mathcal{T}$-Mult and the full subcategory of $\mathcal{T}$-Op consisting of the normalised $\mathcal{T}$-operads.

DEFINITION 7.1. An endofunctor A of $\widehat{\mathbb{G}}$ is *normalised* when for all $X \in \widehat{\mathbb{G}}$, $\{AX\}_0 {\cong} X_0$. A monad (A, η, μ) is normalised when A is normalised as an endofunctor, a cartesian transformation $\alpha : A{\rightarrow}\mathcal{T}$ is called a *normalised collection* when A is normalised, and a $\mathcal{T}$-operad $\alpha : A{\rightarrow}\mathcal{T}$ is normalised when A is normalised as a monad or endofunctor. We shall denote by $\mathcal{T}$-Coll$_0$ the full subcategory of LraEnd$(\widehat{\mathbb{G}})/\mathcal{T}$ consisting of the normalised collections, and by $\mathcal{T}$-Op$_0$ the full subcategory of $\mathcal{T}$-Op consisting of the normalised operads.

A 0-cell of $\mathcal{T}X$ is a pair $(p \in \mathrm{Tr}_0, x : p{\rightarrow}X)$, but then $p = 0$ and by the yoneda lemma we can regard x as an element of X_0. Thus $\mathcal{T}$ is normalised. The category $\mathcal{T}$-Coll$_0$ inherits a strict monoidal structure from LraEnd$(\widehat{\mathbb{G}})/\mathcal{T}$, and the category of monoids therein is exactly $\mathcal{T}$-Op$_0$. We shall allow a very convenient abuse of notation and language: for normalised A write $\{AX\}_0 {=} X_0$ rather than acknowledging the bijection, and speak of X and AX as having the *same* 0-cells. This abuse is justified because for any normalised A, one can obviously redefine A to A' which is normalised in this strict sense, and the assignment $A \mapsto A'$ is part of an equivalence of categories between normalised endofunctors and "strictly normalised endofunctors", regarded as full subcategories of End$(\widehat{\mathbb{G}})$.

We begin by recalling and setting up some notation. Recall how a finite sequence $(X_1, ..., X_k)$ of globular sets may be regarded as a globular set: the set of 0-cells is

$$[k]_0 = \{0, ..., k\},$$

$(X_1, ..., X_k)(i - 1, i) = X_i$ and all the other homs are empty. Since we shall use these sequences often thoughout this section it is necessary to be careful with the use of round brackets with globular sets. For instance X and (X) are different, and so for an endofunctor A of $\widehat{\mathbb{G}}$, one cannot identify AX and $A(X)$!! Observe also that the 0-cell map of a morphism

$$f : (X_1, ..., X_m) \rightarrow (Y_1, ..., Y_n)$$

must be distance preserving, that is it sends consecutive elements to consecutive elements, whenever all the X_i are non-empty globular sets. We regard sequences $(x_0, ..., x_k)$ of 0-cells of a globular set X as maps $x : [k]_0 {\rightarrow} X$ in $\widehat{\mathbb{G}}$. Given any such x we shall define

$$x^* X := (X(x_{i-1}, x_i) : 1{\leq}i{\leq}k),$$

and a map $\overline{x} : x^*X \to X$ of globular sets. The maps $\overline{x}$ and x agree on 0-cells, and $\overline{x}_{i-1,i} = \mathrm{id}$ for $1 \le i \le k$ specifies the hom maps of $\overline{x}$.

Fundamental to this section is the description of the homs of $\mathcal{T}X$ given in lemma(6.12). We shall now refine this and see that an analogous lemma holds for any normalised collection. For X a globular set and a and $b \in X_0$, we shall now understand the hom $\{\mathcal{T}X\}(a,b)$. An n-cell of $\{\mathcal{T}X\}(a,b)$ is a pair (p,f) where $p \in \mathrm{Tr}_{n+1}$ and $f : p \to X$, such that $f\sigma = a$ and $f\tau = b$. First we consider the case $X = (X_1, ..., X_k)$ for globular sets X_i. Writing $p = (p_1, ..., p_m)$ where the $p_i \in \mathrm{Tr}_n$, notice that f_0 must be distance preserving. There will be no such f when $a > b$, and in the case $a \le b$ an n-cell of $\{\mathcal{T}X\}(a,b)$ consists of $p_i \in \mathrm{Tr}_n$ where $a < i \le b$ together with $f_i : p_i \to X_i$. In particular note that when $a = 0$ and $b = k$, $f_0 = \mathrm{id}$. We record this in the following result.

LEMMA 7.2. *Let* $X = (X_1, ..., X_k)$ *in* $\widehat{\mathbb{G}}$. *Then for* $0 \le a, b \le k$ *we have*

$$\{\mathcal{T}X\}(a,b) = \begin{cases} \emptyset & a > b \\ \prod_{a < i \le b} \mathcal{T}X_i & a \le b \end{cases}$$

In particular $\mathcal{T}^{\times} X_i = \{\mathcal{T}X\}(0,k)$.

Now take X to be an arbitrary globular set. Writing $x : [m]_0 \to X$ for the sequence of 0-cells of X defined by f_0, notice that $f : p \to X$ factors uniquely as

$$p \xrightarrow{\ f'\ } x^*X \xrightarrow{\ \overline{x}\ } X$$

and so defines $(p, f') \in \mathcal{T}x^*X_n$ which gets sent to (p,f) by $\mathcal{T}\overline{x}$. Notice that f' is the identity on 0-cells, which is to say that (p,f') is an n-cell of $\{\mathcal{T}x^*X\}(0,m)$. Therefore an n-cell ϕ of $\{\mathcal{T}X\}(a,b)$ is determined uniquely by the following data: $m \in \mathbb{N}$, $x : [m]_0 \to X$ such that $x0 = a$ and $xm = b$, and an n-cell ϕ' of $\{\mathcal{T}x^*X\}(0,m)$. One recovers ϕ from this data by $\{\mathcal{T}\overline{x}\}_{0,m}\phi' = \phi$. Notice also that if any of the homs $X(x(i-1), x_i)$ is empty, then since $\mathcal{T}$ preserves the initial object, one has that $\{\mathcal{T}x^*X\}(0,m)$ is empty by lemma(7.2). Thus one can also specify an n-cell ϕ of $\{\mathcal{T}X\}(a,b)$ uniquely by giving (m, ϕ', x) as above, with the additional condition on x that the homs $X(x(i-1), xi)$ be non-empty for all $1 \le i \le k$. This last condition amounts to saying that one can factor x as

$$[m]_0 \xrightarrow{\ i\ } [m] \longrightarrow X$$

where i is the inclusion of the vertices of $[m]$. We shall call the sequences x satisfying this condition *connected*. We have proved the following refinement of lemma(6.12).

LEMMA 7.3. *Let* X *be a globular set and* a *and* $b \in X_0$.
(1) *The maps*

$$\{\mathcal{T}\overline{x}\}_{0,m} : \{\mathcal{T}x^*X\}(0,m) \to \{\mathcal{T}X\}(a,b)$$

for all $m \in \mathbb{N}$ *and all sequences* $x : [m]_0 \to X$ *such that* $x0 = a$ *and* $xm = b$, *form a coproduct cocone.*
(2) *The maps* $\{\mathcal{T}\overline{x}\}_{0,m}$ *for all* $m \in \mathbb{N}$ *and all connected sequences* $x : [m]_0 \to X$ *such that* $x0 = a$ *and* $xm = b$, *form a coproduct cocone.*

Now for a normalised collection $\alpha : A \to \mathcal{T}$, the extensivity of $\widehat{\mathbb{G}}$ and the cartesianness of α enables us to lift our understanding of the homs of $\mathcal{T}X$ expressed in the

previous two lemmas, to an understanding of the homs of AX. In order to do this in lemma(7.5) below, we require a basic lemma regarding pullbacks and homs in $\widehat{\mathbb{G}}$.

LEMMA 7.4. *Given a commutative square (I)*

$$
\begin{array}{ccc}
W & \xrightarrow{\ h\ } & X \\
{\scriptstyle f}\downarrow & \quad\text{I}\quad & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\ k\ } & Z
\end{array}
\qquad
\begin{array}{ccc}
W(a,b) & \xrightarrow{\ h_{a,b}\ } & X(ha,hb) \\
{\scriptstyle f_{a,b}}\downarrow & \quad\text{II}\quad & \downarrow{\scriptstyle g_{ha,hb}} \\
Y(a,b) & \xrightarrow[\ k_{a,b}\ ]{} & Z(ha,hb)
\end{array}
$$

in $\widehat{\mathbb{G}}$ such that f_0 and g_0 are identities, one has for each $a,b \in W_0$ commuting squares (II) as in the previous display. The square (I) is a pullback iff for all $a,b \in W_0$, the square (II) is a pullback.

PROOF. Suppose that (I) is a pullback and $a,b \in W_0$. Let $y \in Y(a,b)_n$ and $x \in X(ha,hb)_n$ such that $ky = gx$. Then there is a unique $w \in W_{n+1}$ such that $fw = y$ and $hw = x$, and since $f_0 = \mathrm{id}$ and its components commute with sources and targets, one has $w \in W(a,b)_n$ whence (II) is a pullback. Conversely suppose that (II) is a pullback for all $a,b \in W_0$. In dimension 0 (I) is a pullback since f_0 and g_0 are identities. For $n \in \mathbb{N}$ let $y \in Y_{n+1}$ and $x \in X_{n+1}$ such that $ky = gx$. Put $a = s_0 y$ and $b = t_0 b$ so that $y \in Y(a,b)_n$. Since the components of maps in $\widehat{\mathbb{G}}$ commute with sources and targets we have $x \in X(ha,hb)_n$, and since (II) is a pullback there is a unique $w \in W(a,b)_n$ such that $fw = y$ and $hw = x$. Any $w' \in W_{n+1}$ such that $fw' = y$ and $hw' = x$ is in $W(a,b)_n$ since the components of f commute with sources and targets, and so $w' = w$. $\qquad\square$

LEMMA 7.5. *Fix a choice of initial object $\emptyset$ and pullbacks in $\widehat{\mathbb{G}}$, such that the pullback of an identity arrow is an identity. Let $\alpha : A \to \mathcal{T}$ be a normalised collection.*

(1) *Let $X = (X_1, ..., X_k)$ in $\widehat{\mathbb{G}}$. Then for $0 \le a, b \le k$ we have*

$$
\{AX\}(a,b) = \begin{cases} \emptyset & a > b \\ \{Ax^*X\}(0, b-a) & a \le b \end{cases}
$$

where $x : [b-a]_0 \to X$ is given by $xi = a + i$.

(2) *The maps*

$$
\{A\overline{x}\}_{0,m} : \{Ax^*X\}(0,m) \to \{AX\}(a,b)
$$

for all $m \in \mathbb{N}$ and all sequences $x : [m]_0 \to X$ such that $x0 = a$ and $xm = b$, form a coproduct cocone.

(3) *The maps $\{A\overline{x}\}_{0,m}$ for all $m \in \mathbb{N}$ and all connected sequences $x : [m]_0 \to X$ such that $x0 = a$ and $xm = b$, form a coproduct cocone.*

PROOF. In the case $X = (X_1, ..., X_k)$ with $a > b$ one has $\{\alpha_X\}_{a,b} : \{AX\}(a,b) \to \emptyset$ by lemma(7.2), and since the initial object of $\widehat{\mathbb{G}}$ is strict, one has $\{AX\}(a,b) = \emptyset$. Given any X and $x : [m]_0 \to X$ such that $x0 = a$ and $xm = b$, we have that

$$
\begin{array}{ccc}
\{Ax^*X\}(0,m) & \xrightarrow{\{A\overline{x}\}_{0,m}} & \{AX\}(a,b) \\
{\scriptstyle \{\alpha_{x^*X}\}_{0,m}}\downarrow & & \downarrow{\scriptstyle \{\alpha_X\}_{a,b}} \\
\{\mathcal{T}x^*X\}(0,m) & \xrightarrow[\{\mathcal{T}\overline{x}\}_{0,m}]{} & \{\mathcal{T}X\}(a,b)
\end{array}
$$

is a pullback by lemma(7.4) and the cartesianness of α. In the case $X = (X_1, ..., X_k)$ with $a \le b$ and $x : [b-a]_0 \to X$ given by $xi = a + i$, $\{\mathcal{T}\overline{x}\}_{0,b-a}$ is the identity by lemma(7.2), thus so is $\{A\overline{x}\}_{0,b-a}$ and we have proved (1). In the general case considering all $m \in \mathbb{N}$ and sequences (resp. connected sequences) $x : [m]_0 \to X$ with $x0 = a$ and $xm = b$, the $\{\mathcal{T}\overline{x}\}_{0,m}$ form a coproduct cocone by lemma(7.3), and thus so do the $\{A\overline{x}\}_{0,m}$ by extensivity, which gives (2) and (3). $\qquad\square$

For a normalised collection A, $k \in \mathbb{N}$ and $X_i \in \widehat{\mathbb{G}}$ where $1 \le i \le k$, define

$$\overline{\underset{i}{A}} X_i = \{AX\}(0,k)$$

where $X = (X_1, ..., X_k)$.

REMARK 7.6. There is an analogy between lemma(7.5) and the Lagrangian formulation of quantum mechanics. In this analogy one regards any globular set X, to which one would apply a collection, as a *state space* the 0-cells of which are called *states*. A normalised collection A is then a *type of quantum mechanical process*, with the hom $\{AX\}(a,b)$ playing the role of the *amplitude* that the process starts in state a and finishes in state b. The *basic* amplitudes are the $\{AX\}(0,k)$ where $X = (X_1, ..., X_k)$. In terms of these analogies, lemma(7.5) expresses the sense in which the general amplitude $\{AX\}(a,b)$ may be regarded as the sum of the basic amplitudes over all the "paths" between a and b, that is, as a sort of discrete Feynman integral. The formula just given expresses this passage between basic and general amplitudes as a particular strong monoidal functor, which allows us to view normalised operads as multitensors, and algebras of such an operad as categories enriched in the corresponding multitensor.

The reader should be aware that it first became apparent to the authors that lemma(7.5) is fundamental to the proof of theorems(7.7) and (7.8), and the above analogy was noticed afterwards.

THEOREM 7.7. *The assignment $A \mapsto \overline{A}$ is the object map of a strong monoidal functor*

$$\overline{(\;)} : \mathcal{T}\text{-Coll}_0 \to \text{Dist}(\widehat{\mathbb{G}}).$$

For a normalised operad A, one has an isomorphism $A\text{-Alg} \cong \overline{A}\text{-Cat}$ commuting with the forgetful functors into Set.

PROOF. The above definition is clearly functorial in the X_i so one has $\overline{A} : \mathcal{M}\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$. A morphism of normalised collections $\phi : A \to B$ is a cartesian transformation between A and B, and such a ϕ then induces a natural transformation $\overline{\phi} : \overline{A} \to \overline{B}$ by the formula $\overline{\phi}_{X_i} = \{\phi_X\}_{0,k}$. The cartesianness of ϕ and lemma(7.4) ensures that $\overline{\phi}$ is cartesian. In particular $\overline{\mathcal{T}} = \mathcal{T}^\times$ by lemma(7.2) and so for a given normalised collection $\alpha : A \to \mathcal{T}$, one obtains a cartesian $\overline{\alpha} : \overline{A} \to \mathcal{T}^\times$. Now by example(3.2) $\mathcal{T}^\times$ is distributive (ie preserves coproducts in each variable) and so $\overline{A}$ is also because of the cartesianness of $\overline{\alpha}$ and the stability of coproducts in $\widehat{\mathbb{G}}$. The assignment $\phi \mapsto \overline{\phi}$ described above is clearly functorial, and so $\overline{(\;)}$ is indeed well-defined as a functor into $\text{Dist}(\widehat{\mathbb{G}})$.

Since $X(0,k)$ is empty when $k \ne 1$ and just X_1 when $k = 1$, we have $\overline{1} = I$ the unit of $\text{Dist}(\widehat{\mathbb{G}})$. Let A and B be normalised collections and $X = (X_1, ..., X_m)$. By lemma(7.5) the morphisms

$$\{A\overline{x}\}_{0,k} : \{Ax^*BX\}(0,k) \to \{ABX\}(0,m)$$

where $k \in \mathbb{N}$ and $x : [k]_0 \to BX$ such that $x0 = 0$ and $xk = m$, form a coproduct cocone. By the definition of the tensor product in $\mathrm{Dist}(\widehat{\mathbb{G}})$, this induces an isomorphism $\overline{AB} \cong \overline{A} \circ \overline{B}$. We now argue that these isomorphisms satisfy the coherence conditions of a strong monoidal functor. Recall that the tensor product in $\mathrm{Dist}(\widehat{\mathbb{G}})$ is defined using coproducts. A different choices of coproducts give rise to different monoidal structures on $\mathrm{Dist}(\widehat{\mathbb{G}})$, though for two such choices the identity functor on $\mathrm{Dist}(\widehat{\mathbb{G}})$ inherits unique coherence isomorphisms that make it strong monoidal and thus an isomorphism of monoidal categories. Because of this one may easily check that if a given strong monoidal coherence diagram commutes for a particular choice of defining coproducts of the monoidal structure of $\mathrm{Dist}(\widehat{\mathbb{G}})$, then this diagram commutes for any such choice. Thus to verify a given strong monoidal coherence diagram, it suffices to see that it commutes for some choice of coproducts. But for any such diagram one can simply choose the coproducts so that all the coherence isomorphisms involved in *just* that diagram are identities. Note that this is *not* the same as specifying $\mathrm{Dist}(\widehat{\mathbb{G}})$'s monoidal structure so as to make $\overline{(\ )}$ strict monoidal. This finishes the proof that $\overline{(\ )}$ is strong monoidal.

Let A be a normalised operad and Z be a set. To give a globular set X with $X_0 = Z$ and $x : AX \to X$ which is the identity on 0-cells, is to give globular sets $X(y, z)$ for all $y, z \in Z$ and maps $x_{y,z} : \{AX\}(y, z) \to X(y, z)$. By lemma(7.5) the $x_{y,z}$ amount to giving for each $k \in \mathbb{N}$ and $f : [k]_0 \to X$ such that $f0 = y$ and $fk = z$, a map

$$x_f : \overline{\mathrm{A}}_i X(f_{i-1}, fi) \to X(y, z)$$

since $\overline{\mathrm{A}}_i X(f_{i-1}, fi) = \{Af^*X\}(0, k)$, that is $x_f = x_{y,z}\{A\overline{f}\}_{0,k}$. For $y, z \in Z$, one has a unique $f : [1]_0 \to X$ given by $f0 = y$ and $f1 = z$. The naturality square for η at $\overline{f}$ implies that $\{\eta_X\}_{y,z} = \{A\overline{f}\}_{0,1}\{\eta_{(X(y,z))}\}_{0,1}$ and the definition of $\overline{(\ )}$ says that $\{\eta_{(X(y,z))}\}_{0,1} = \overline{\eta}_{X(y,z)}$. Thus to say that a map $x : AX \to X$ satisfies the unit law of an A-algebra is to say that x is the identity on 0-cells and that the x_f described above satisfy the unit axioms of an $\overline{A}$-category.

To say that x satisfies the associative law is to say that for all $y, z \in Z$,

(5)
$$\{A^2X\}(y, z) \xrightarrow{\{\mu_X\}_{y,z}} \{AX\}(y, z)$$
$$\{Ax\}_{y,z} \downarrow \qquad\qquad\qquad \downarrow x_{y,z}$$
$$\{AX\}(y, z) \xrightarrow{\quad x_{y,z} \quad} X(y, z)$$

commutes. Given $f : [m]_0 \to X$ with $f0 = y$ and $fm = z$, and $g : [k]_0 \to Af^*X$ with $g0 = 0$ and $gk = m$, precomposing (5) with the composite map

(6)
$$\{Ag^*Af^*X\}(0, k) \xrightarrow{\{A\overline{g}\}_{0,k}} \{A^2f^*X\}(0, m) \xrightarrow{\{A\overline{f}\}_{0,m}} \{A^2X\}(y, z)$$

and using lemma(7.5) one can see that one obtains the commutativity of

(7)
$$\overline{\mathrm{A}}_i \overline{\mathrm{A}}_j X(f((i, j) - 1), f(i, j)) \xrightarrow{\overline{\mu}} \overline{\mathrm{A}}_{ij} X(f((i, j) - 1), f(i, j))$$
$$\overline{\mathrm{A}}_i x_{\{A\overline{f}\}_g} \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow x_f$$
$$\overline{\mathrm{A}}_i X(g(i-1), gi) \xrightarrow{\qquad\qquad x_g \qquad\qquad} X(y, z)$$

where $1 \leq i \leq k$, $1 \leq j \leq m_i$, with the m_i determined in the obvious way by g. That is, the associative law for x, namely (5), implies the $\overline{A}$-category associative laws (7). Conversely since the composites (6) over all choices of f and g form a coproduct cocone by lemma(7.5), (7) also implies (5). This completes the description of the object part of A-Alg $\cong \overline{A}$-Cat.

Let (X, x) and (X', x') be A-algebras and $F_0 : X_0 \to X'_0$ be a function. To give $F : X \to X'$ with 0-cell map F_0 is to give for all $y, z \in X_0$, maps $F_{y,z} : X(y, z) \to X'(F_0 y, F_0 z)$. By lemma(7.5) to say that F is an algebra map is equivalent to saying that F_0 and the $F_{y,z}$ form an $\overline{A}$-functor. The isomorphism A-Alg $\cong \overline{A}$-Cat just described commutes with the forgetful functors into Set by definition. $\qquad \square$

Early in the above proof we saw that $\overline{(\ )}$ sends morphisms in $\mathcal{T}$-Coll$_0$ to cartesian transformations. Since $\mathcal{T}^\times$ is tight by proposition(6.11) and lemma(6.10), this implies by theorem(7.7) that $\overline{(\ )}$ may in fact be regarded as a strong monoidal functor

$$\overline{(\ )} : \mathcal{T}\text{-Coll}_0 \to \text{LraDist}(\widehat{\mathbb{G}})/\mathcal{T}^\times.$$

For this manifestation of $\overline{(\ )}$ we have the following result.

THEOREM 7.8. *The functor $\overline{(\ )}$ just described is an equivalence of categories $\mathcal{T}$-Coll$_0 \simeq \text{LraDist}(\widehat{\mathbb{G}})/\mathcal{T}^\times$.*

PROOF. We will verify that $\overline{(\ )}$ is essentially surjective on objects and fully faithful. For a cartesian $\varepsilon : E \to \mathcal{T}^\times$ we now define $\alpha : A \to \mathcal{T}$ so that $\overline{\alpha} \cong \varepsilon$. For $X \in \widehat{\mathbb{G}}$ define $\{AX\}_0 = X_0$, and for $x, y \in X_0$, define $\{AX\}(x, y)$ as a coproduct with coproduct injections

$$c_f : \underset{i}{\mathrm{E}}\, X(f(i-1), fi) \to \{AX\}(x, y)$$

for each $f : [k]_0 \to X$ with $f0 = x$ and $fk = y$. This definition is functorial in X in the obvious way. The components of α are identities on 0-cells with the hom maps determined by the commutativity of

$$(8) \qquad \begin{array}{ccc} \underset{i}{\mathrm{E}}\, X(f(i-1), fi) & \xrightarrow{\ c_f\ } & \{AX\}(x, y) \\ {\scriptstyle \varepsilon} \downarrow & & \downarrow {\scriptstyle \{\alpha_X\}_{x,y}} \\ \{\mathcal{T}f^*X\}(0, k) & \underset{\{\mathcal{T}\overline{f}\}_{0,k}}{\longrightarrow} & \{\mathcal{T}X\}(x, y) \end{array}$$

for all f as above. Since $\widehat{\mathbb{G}}$ is extensive these squares are pullbacks, and so by lemma(7.4) α defined in this way is indeed cartesian. In the case where $X = (X_1, ..., X_k)$ and f is the identity on 0-cells, one has $\{\mathcal{T}\overline{f}\}_{0,k} = \text{id}$ and so (8) gives $\overline{\alpha} \cong \varepsilon$ as required. To verify fully faithfulness let $\alpha : A \to \mathcal{T}$ and $\beta : B \to \mathcal{T}$ be normalised collections, and $\phi : \overline{A} \to \overline{B}$ be a cartesian transformation. To finish the proof it suffices, by the tightness of $\mathcal{T}$ and $\mathcal{T}^\times$ and lemma(6.8), to define a cartesian transformation $\psi : A \to B$ unique such that $\overline{\psi} = \phi$. For $X \in \widehat{\mathbb{G}}$ and $f : [k]_0 \to X$ this last equation says that such a ψ must satisfy

$$\{\psi_{f^*X}\}(0, k) = \phi_{X(f(i-1), fi)} : \{Af^*X\}(0, k) \to \{Bf^*X\}(0, k).$$

The cartesianness of ψ and the tightness of $\mathcal{T}$ implies $\psi\beta = \alpha$ by lemma(6.8), and so $\{\psi_X\}_0$ is the identity. For $x, y \in X_0$ the map $\{\psi_X\}_{x,y}$ is determined by the

commutativity of

$$
\begin{array}{ccc}
\{Af^*X\}(0,k) & \xrightarrow{\{A\overline{f}\}_{0,k}} & \{AX\}(x,y) \\
{\scriptstyle \{\psi_{f^*X}\}(0,k)}\downarrow & & \downarrow{\scriptstyle \{\psi_X\}_{x,y}} \\
\{Bf^*X\}(0,k) & \xrightarrow[\{B\overline{f}\}_{0,k}]{} & \{BX\}(x,y)
\end{array}
$$

for all f, since the $\{A\overline{f}\}_{0,k}$ form a coproduct cocone by lemma(7.5). Note also that this square is a pullback by the extensivity of $\widehat{\mathbb{G}}$. This completes the definition of the components of ψ and the proof that they are determined uniquely by ϕ and the equation $\overline{\psi} = \phi$, and so to finish the proof one must verify that the ψ_X are cartesian natural in X. To this end let $F : X \to Y$. Since the components of α are identities in dimension 0 it suffices by lemma(7.4) to show that for all $x, y \in X_0$ the squares

$$
\text{(9)} \qquad
\begin{array}{ccc}
\{AX\}(x,y) & \xrightarrow{\{AF\}_{x,y}} & \{AY\}(F_0x, F_0y) \\
{\scriptstyle \{\psi_X\}_{x,y}}\downarrow & & \downarrow{\scriptstyle \{\psi_Y\}_{F_0x, F_0y}} \\
\{BX\}(x,y) & \xrightarrow[\{BF\}_{x,y}]{} & \{BY\}(F_0x, F_0y)
\end{array}
$$

are pullbacks. For all $f : [k]_0 \to X$ one has $F\overline{f} = \overline{Ff}$ by definition, and so the composite square

$$
\begin{array}{ccccc}
\{Af^*X\}(0,k) & \xrightarrow{\{A\overline{f}\}_{0,k}} & \{AX\}(x,y) & \xrightarrow{\{AF\}_{x,y}} & \{AY\}(F_0x, F_0y) \\
{\scriptstyle \{\psi_{f^*X}\}(0,k)}\downarrow & & {\scriptstyle \{\psi_X\}_{x,y}}\downarrow & & \downarrow{\scriptstyle \{\psi_Y\}_{F_0x, F_0y}} \\
\{Bf^*X\}(0,k) & \xrightarrow[\{B\overline{f}\}_{0,k}]{} & \{BX\}(x,y) & \xrightarrow[\{BF\}_{x,y}]{} & \{BY\}(F_0x, F_0y)
\end{array}
$$

is a pullback, and so by the extensivity of $\widehat{\mathbb{G}}$ (9) is indeed a pullback since the $\{A\overline{f}\}_{0,k}$ for all f form a coproduct cocone. $\qquad\square$

REMARK 7.9. The equivalence of theorem(7.8) could have been described differently. This alternative view involves the adjoint endofunctors D and Σ of $\widehat{\mathbb{G}}$. For $X \in \widehat{\mathbb{G}}$, DX is obtained by discarding the 0-cells and putting $\{DX\}_n = X_{n+1}$, ΣX has one 0-cell and $\Sigma X_{n+1} = X_n$ and one has $D \dashv \Sigma$. The effect of D and Σ on arrows provides an adjunction

$$
\text{(10)} \qquad
\widehat{\mathbb{G}}/\mathcal{T}1 \underset{\Sigma_{D\mathcal{T}1}}{\overset{D_{\mathcal{T}1}}{\rightleftarrows}} \widehat{\mathbb{G}}/D\mathcal{T}1 \;,
$$

and the right adjoint $\Sigma_{D\mathcal{T}1}$ is fully faithful since Σ is. Thus (10) restricts to an equivalence between the full subcategory $\mathcal{N}$ of $\widehat{\mathbb{G}}/\mathcal{T}1$ consisting of those $f : X \to \mathcal{T}1$ such that X_0 is singleton. Evaluating at 1 gives an equivalence between $\mathcal{T}\text{-Coll}_0$ and the full subcategory of $\widehat{\mathbb{G}}/\mathcal{T}1$ just described. By evaluating at 1 and by the definitions of D and $\mathcal{T}1$ one obtains $\text{LraDist}(\widehat{\mathbb{G}})/\mathcal{T}^{\times} \simeq \widehat{\mathbb{G}}/D\mathcal{T}1$. Finally these

equivalences fit together into a square

$$
\begin{array}{ccc}
\mathcal{T}\text{-Coll}_0 & \xrightarrow{\overline{(\,)}} & \mathrm{LraDist}(\widehat{\mathbb{G}})/\mathcal{T}^{\times} \\
{\scriptstyle \mathrm{ev}_1} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_1} \\
\mathcal{N} & \xrightarrow[D_{\mathcal{T}1}]{} & \widehat{\mathbb{G}}/D\mathcal{T}1
\end{array}
$$

which one may easily verify commutes up to isomorphism. These equivalences ev_1 really just express the equivalence of two different ways of viewing collections and their multitensorial analogues, and so modulo this, the equivalence from (10) expresses in perhaps more concrete terms what $\overline{(\,)}$ does. However we have chosen to work with $\overline{(\,)}$ because this point of view makes clearer the relationship between algebras and enriched categories that we have expressed in theorem(7.7).

Putting together theorem(7.8) and theorem(5.9) one obtains the equivalence between normalised $\mathcal{T}$-operads, $\mathcal{T}$-multitensors and $M\mathcal{T}$-operads.

COROLLARY 7.10. $\mathcal{T}\text{-Op}_0 \simeq \mathcal{T}\text{-Mult} \simeq M\mathcal{T}\text{-Op}$.

8. Finite dimensions and the algebras of $\mathcal{T}$

We shall now explain how the results of this paper specialise to finite dimensions, and show how one can see that the algebras of $\mathcal{T}$ really are strict ω-categories defined in the usual way by successive enrichment.

The category $\mathbb{G}_{\leq n}$ is defined to be the full subcategory of $\mathbb{G}$ consisting of the $k \in \mathbb{N}$ such that $0 \leq k \leq n$. The objects of $\widehat{\mathbb{G}}_{\leq n}$ are called n-globular sets. By definition the monad $\mathcal{T}$ on $\widehat{\mathbb{G}}$ restricts to n-globular sets: the description of $\mathcal{T}X_n$ depends only on the k-cells of X for $k \leq n$. Thus one has a monad $\mathcal{T}_{\leq n}$ on $\widehat{\mathbb{G}}_{\leq n}$. Our description of $\mathcal{T}$ from section(6) restricts also, and so the monads $\mathcal{T}_{\leq n}$ are l.r.a, coproduct preserving and tight. In fact, by direct inspection, everything we have done in this paper that has anything to do with $\mathcal{T}$ restricts to finite dimensions.

In particular for $n \in \mathbb{N}$, denoting by $\mathcal{T}_{\leq 1+n}\text{-Coll}_0$ the category of normalised $(1+n)$-collections, whose objects are cartesian transformations $\alpha : A \to \mathcal{T}_{\leq 1+n}$ whose components are identities in dimension 0, one has a functor

$$\overline{(\,)} : \mathcal{T}_{\leq 1+n}\text{-Coll}_0 \to \mathrm{Dist}(\widehat{\mathbb{G}}_{\leq n})$$

whose object map is given by the formula

$$\overline{\underset{i}{\mathrm{A}}} X_i = \{AX\}(0, k)$$

where A is a normalised $(1 + n)$-collection, $k \in \mathbb{N}$ and $X_i \in \widehat{\mathbb{G}}_{\leq n}$ where $1 \leq i \leq k$, and $X \in \widehat{\mathbb{G}}_{\leq 1+n}$ is defined as $X = (X_1, ..., X_k)$. The finite dimensional analogue of theorem(7.7) is

THEOREM 8.1. *The functor $\overline{(\,)}$ just described is a strong monoidal functor, and for a normalised $(1 + n)$-operad A, one has an isomorphism $A\text{-Alg} \cong \overline{A}\text{-Cat}$ commuting with the forgetful functors into* Set.

As before one may also regard $\overline{(\,)}$ as a strong monoidal functor

$$\overline{(\,)} : \mathcal{T}_{\leq 1+n}\text{-Coll}_0 \to \mathrm{LraDist}(\widehat{\mathbb{G}}_{\leq n})/\mathcal{T}_{\leq n}^{\times}.$$

and the analogue of theorem(7.8) is

THEOREM 8.2. *The functor* $\overline{(\,)}$ *just described is an equivalence of categories* $\mathcal{T}_{\leq 1+n}\text{-Coll}_0 \simeq \text{LraDist}(\widehat{\mathbb{G}}_{\leq n})/\mathcal{T}_{\leq n}^{\times}.$

and so we have

COROLLARY 8.3. $\mathcal{T}_{\leq 1+n}\text{-Op}_0 \simeq \mathcal{T}_{\leq n}\text{-Mult} \simeq M\mathcal{T}_{\leq n}\text{-Op}.$

One can think of n as an ordinal instead of a natural number, and then the original results from section(7) correspond to the case $n = \omega$.

All along we have been working with the monads $\mathcal{T}_{\leq n}$ as formally defined combinatorial objects. Given the results of this paper however, it is now easy to see that their algebras are indeed strict n-categories. The usual definition of strict n-categories is by successive enrichment. One defines 0-Cat = Set and $(1 + n)$-Cat = $(n$-Cat)-Cat for $n \in \mathbb{N}$ where n-Cat is regarded as monoidal via cartesian product. Recasting this a little more formally, $(-)$-Cat is an endofunctor of the full subcategory of CAT consisting of categories with finite products. Writing 0 for the terminal object of this category, that is the terminal category, one has by functoriality a sequence

$$\cdots \longrightarrow 3\text{-Cat} \longrightarrow 2\text{-Cat} \longrightarrow 1\text{-Cat} \longrightarrow 0\text{-Cat} \longrightarrow 0$$

Explicitly the maps in this diagram are the obvious forgetful functors. The limit of this diagram is formed as in CAT, and provides the definition of the category ω-Cat. Then by theorem(7.7) and proposition(2.8) we have isomorphisms

$$\phi_n : \mathcal{T}_{\leq 1+n}\text{-Alg} \rightarrow (\mathcal{T}_{\leq n}\text{-Alg})\text{-Cat}$$

Let us write Enr for the endofunctor $\mathcal{V} \mapsto \mathcal{V}$-Cat that we have just been considering. The isomorphisms ϕ_n are natural in the sense of the following lemma, which enables us to then formally identify the algebras of $\mathcal{T}$ in theorem(8.5).

LEMMA 8.4. *For* $n \in \mathbb{N}$ *let* $\text{tr}_n : \mathcal{T}_{\leq 1+n}\text{-Alg} \rightarrow \mathcal{T}_{\leq n}\text{-Alg}$ *be the forgetful functor given by truncation. The square*

$$
\begin{array}{ccc}
\mathcal{T}_{\leq 2+n}\text{-Alg} & \xrightarrow{\;\text{tr}_{1+n}\;} & \mathcal{T}_{\leq 1+n}\text{-Alg} \\
{\scriptstyle\phi_{1+n}}\big\downarrow & & \big\downarrow{\scriptstyle\phi_n} \\
(\mathcal{T}_{\leq 1+n}\text{-Alg})\text{-Cat} & \xrightarrow[\text{Enr}(\text{tr}_n)]{} & (\mathcal{T}_{\leq n}\text{-Alg})\text{-Cat}
\end{array}
$$

commutes for all $n \in \mathbb{N}$.

PROOF. One obtains ϕ_n explicitly as the composite of two isomorphisms

$$\mathcal{T}_{\leq 1+n}\text{-Alg} \rightarrow \mathcal{T}_{\leq n}^{\times}\text{-Cat} \rightarrow (\mathcal{T}_{\leq n}\text{-Alg})\text{-Cat}$$

the first of which is described explicitly in the proof of theorem(7.7), and the second in the proof of proposition(2.8), and using these descriptions one may easily verify directly the desired naturality. $\square$

THEOREM 8.5. *For* $0 \leq n \leq \omega$, $\mathcal{T}_{\leq n}\text{-Alg} \cong n\text{-Cat}$.

PROOF. Write $t : 0\text{-Cat} \rightarrow 0$ for the unique functor. By the definition of ω-Cat it suffices to provide isomorphisms $\psi_n : \mathcal{T}_{\leq n}\text{-Alg} \rightarrow n\text{-Cat}$ for $n \in \mathbb{N}$ natural in the

sense that

$$\mathcal{T}_{\leq 1+n}\text{-Alg} \xrightarrow{\ \mathrm{tr}_n\ } \mathcal{T}_{\leq n}\text{-Alg}$$

$$\psi_{1+n} \downarrow \qquad\qquad \downarrow \psi_n$$

$$(1+n)\text{-Cat} \xrightarrow[\mathrm{Enr}^{1+n}(t)]{} n\text{-Cat}$$

commutes for all n. Take $\psi_0 = 1_{\mathrm{Set}}$ and by induction define ψ_{1+n} as the composite

$$\mathcal{T}_{\leq 1+n}\text{-Alg} \xrightarrow{\ \phi_n\ } (\mathcal{T}_{\leq n}\text{-Alg})\text{-Cat} \xrightarrow{\mathrm{Enr}(\psi_n)} (1+n)\text{-Cat} \ .$$

The case $n = 0$ for ψ's naturality comes from the fact that the isomorphisms that comprise ϕ_1 (see lemma(8.4)) are defined over Set. The inductive step follows easily from lemma(8.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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References

[1] M. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics*, 136:39–103, 1998.

[2] M. Batanin. The Eckmann-Hilton argument and higher operads. *Advances in Mathematics*, 217:334–385, 2008.

[3] M. Batanin, D-C. Cisinski, and M. Weber. Algebras of higher operads as enriched categories II. In preparation.

[4] C. Berger. Iterated wreath product of the simplex category and iterated loop spaces. *Advances in Mathematics*, 213:230–270, 2007.

[5] A. Carboni, S. Lack, and R.F.C.Walters. Introduction to extensive and distributive categories. *J. Pure Appl. Algebra*, 84:145–158, 1993.

[6] S. Crans. A tensor product for gray categories. *Theory and applications of categories*, 5:12–69, 1999.

[7] B. Day and R. Street. Lax monoids, pseudo-operads, and convolution. In *Diagrammatic Morphisms and Applications*, volume 318 of *Contemporary Mathematics*, pages 75–96, 2003.

[8] R. Gordon, A.J. Power, and R. Street. *Coherence for tricategories*. Number 117 in Mem. Amer. Math. Soc. AMS, 1995.

[9] G.M. Kelly. *Basic concepts of enriched category theory, LMS lecture note series*, volume 64. Cambridge University Press, 1982.

[10] G.M. Kelly. On the operads of J.P. May. *TAC reprints series*, 13, 2005.

[11] S. Lack. Codescent objects and coherence. *J. Pure Appl. Algebra*, 175:223–241, 2002.

[12] T. Leinster. *Higher operads, higher categories*. lecture note series. London Mathematical Society, 2003.

[13] J. McClure and J. Smith. Cosimplicial objects and little n-cubes I. *Amer. J. Math*, 126:1109–1153, 2004.

[14] M. Weber. Generic morphisms, parametric representations, and weakly cartesian monads. *Theory and applications of categories*, 13:191–234, 2004.

[15] M. Weber. Familial 2-functors and parametric right adjoints. *Theory and applications of categories*, 18:665–732, 2007.

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