The purpose of this talk is to introduce *2-toposes*, which are 2-categories rich enough so that a lot of category theory can be developed inside them. They are one part of a meta-theory whose purpose is to organise the combinatorial machinery of higher dimensional category theory. In particular if you have a 2-topos $\mathcal{K}$ then you can express what it means for an object of $\mathcal{K}$ to have small colimits, and the basic theory of colimits unfolds in nice analogy with $\mathcal{K}$=CAT case.

Overview:

- operadic motivations.

- 2-toposes and examples.

- results on internal cocompleteness.

One can recognise a space X as having the homotopy type of a loop space when there are continuous functions

$$K_n \times X^n \to X$$

respecting certain axioms, where K_n is a convex polytope called the n-th associahedron.

The categorical/combinatorial ingredients for the "classical" theory of operads are:

1. a braided monoidal category $\mathcal{V}$ (which in the above example is Top).

2. an operad in $\mathcal{V}$ which is a sequence

$$K : \mathbb{N} \to \mathcal{V}$$

   of objects of $\mathcal{V}$ together with more data and axioms encoding substitution.

3. K can act on objects

$$X : 1 \to \mathcal{V}$$

   of $\mathcal{V}$ to express structure.

Colimits arise in the classical theory of operads in the following ways:

1. defining the substitution tensor product of collections.

$$(K \circ L)_n = \sum_{n_1+...+n_r=n} K_r \otimes L_{n_1} \otimes ... \otimes L_{n_r}$$

2. turning operads into monads.

$$\overline{K}(X) = \sum_{n \in \mathbb{N}} K_n \otimes X^{\otimes n}$$

3. comparing different types of operads (eg: symmetric versus non-symmetric).

all requiring that the $\mathcal{V}$ you work in has enough colimits, and that these colimits are compatible with $\otimes$.

From the perspective of higher category theory the raw data for a space X is its underlying globular set

$$X_0 \overset{\longleftarrow}{\underset{\longleftarrow}{}} X_1 \overset{\longleftarrow}{\underset{\longleftarrow}{}} X_2 \overset{\longleftarrow}{\underset{\longleftarrow}{}} X_3 \overset{\longleftarrow}{\underset{\longleftarrow}{}} \cdots$$

and the classical

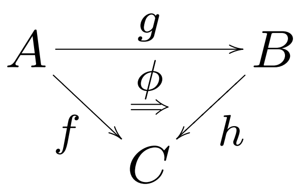
$$K : \mathbb{N} \to \mathcal{V} \qquad X : 1 \to \mathcal{V}$$

is replaced by

$$K : \mathsf{Tr} \to \mathsf{Sp}_{\mathbb{G}}(\mathsf{Set}) \qquad X : 1 \to \mathsf{Sp}_{\mathbb{G}}(\mathsf{Set})$$

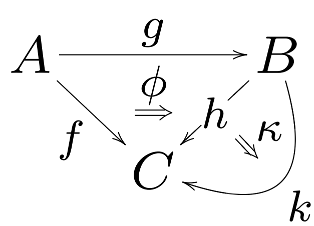
Another important class of examples: take a $\mathcal{V}$ from the classical theory and suspend it to get an n-globular category. **Organising the theory of higher operads takes you into the world of globular categories!**

A 2-cell



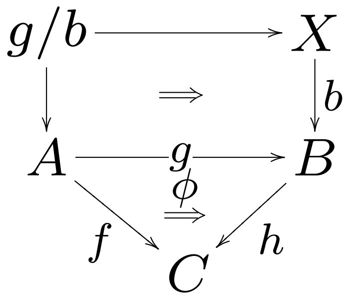
in a 2-category $\mathcal{K}$ exhibits h as a *left extension* of f along g when $\forall k$ pasting with ϕ:

$$\kappa : h \Rightarrow k \qquad \mapsto$$



provides a bijection between 2-cells $h \Rightarrow k$ and 2-cells $f \Rightarrow kg$.

This 2-cell is a *pointwise* left extension when for all $b : X \to B$, the composite



exhibits hb as a left extension.

Roughly then, C is cocomplete when pointwise left extensions into it exist.

So we can ask:

1. Is $\mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$ or more generally $\mathrm{Sp}_{\mathbb{G}}(\mathcal{V})$ where $\mathcal{V}$ is a cocomplete category, cocomplete as a globular category?

2. What about $\Sigma^n(\mathcal{V})$?

3. Is $\mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$ in fact the small globular colimit completion of 1 just as Set is the small colimit completion of 1?

Recall that a category $\mathcal{E}$ is an *elementary topos* when

1. it has finite limits,

2. is cartesian closed, and

3. has a subobject classifier.

So what's a 2-categorical analogue of subobject classifier?

To any functor $f : A \to \mathsf{Set}$ a version of the Grothendieck construction associates to f a discrete opfibration $G(f) : e(f) \to A$ to provide a fully faithful functor

$$G_{U,A} : \mathsf{CAT}(A, \mathsf{Set}) \to \mathsf{DOpFib}(A)$$

whose image consists of those discrete opfibrations with small fibres.

The analogy with topos theory comes about because G can be described as

$$\begin{array}{ccc} e(f) & \longrightarrow & \mathsf{Set}_\bullet \\ {\scriptstyle G(f)}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle U} \\ A & \xrightarrow[f]{} & \mathsf{Set} \end{array}$$

Definition 1 *Let $\mathcal{K}$ be a finitely complete 2-category. A discrete opfibration $\tau : \Omega_\bullet \to \Omega$ in $\mathcal{K}$ is* classifying *when the functors*

$$G_{\tau,A} : \mathcal{K}(A, \Omega) \to \mathsf{DFib}(1, A)$$

are fully faithful for all $A \in \mathcal{K}$.

Definition 2 *A* 2-topos $(\mathcal{K}, (-)^\circ, \tau)$ *is a finitely complete cartesian closed 2-category $\mathcal{K}$ equipped with a duality involution $(-)^\circ$ and a classifying discrete opfibration $\tau : \Omega_\bullet \to \Omega$.*

Theorem 3 *Let*

$$\mathcal{A} \underset{S}{\overset{E}{\rightleftarrows}} \mathcal{B} \quad (E \dashv S)$$

be a 2-adjunction and $\mathcal{A}$ and $\mathcal{B}$ be finitely complete 2-categories. If

1. *E preserves pullbacks, and*

2. *naturality squares for the counit at maps in $\mathcal{A}$ which are discrete opfibrations, are pullbacks*

then S preserves classifying discrete opfibrations.

Examples:

1. The subobject classifier

$$\tau : 1 \to \Omega$$

   of an elementary topos $\mathcal{E}$ regarded as an internal functor is a classifying discrete opfibration in Cat($\mathcal{E}$).

2. U : $\mathrm{Set}_\bullet \to \mathrm{Set}$ is a c.d.o for CAT. Similarly with Set replaced by finite sets.

3. $\mathrm{Sp}_{\mathbb{G}}(U)$, $\Sigma\mathrm{Sp}_{\mathbb{G}}(U)$, $\Sigma^2\mathrm{Sp}_{\mathbb{G}}(U)$, ... are c.d.o's for CAT($\widehat{\mathbb{G}}$).

4. there's a version of $\mathrm{Sp}_{\mathbb{G}}(U)$ for CAT($\widehat{\mathbb{C}}$).

Definition 4 *A map* $f : A \to B$ *in a 2-topos is* admissible *when the discrete fibration* f/B *is in the image of the canonical functor*

$$\mathcal{K}(B, \widehat{A}) \to \mathsf{DFib}(A, B)$$

An object A *is admissible when the map* 1_A *is admissible.* A *is* small *when it is admissible and* $\widehat{A}$ *is admissible.*

There is a general result which enables one to characterise smalls and admissibles. For instance for $\mathsf{Sp}_{\mathbb{G}}(U)$ the corresponding small globular categories are just those that are small at each level.

Definition 5 *An object* A *in a 2-topos is* co-complete *when it admits all pointwise left extensions along any* $h : X \to Z$ *such that* X *is small and* Z *is admissible.*

Theorem 6 *Let*

$$\mathcal{A} \underset{S}{\overset{E}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{B}$$

be a 2-adjunction, $\mathcal{A}$ and $\mathcal{B}$ be 2-toposes which exhibit small lax pullbacks. If

1. *E preserves small objects, admissible objects and lax pullbacks up to a right adjoint section; and*

2. *S preserves admissible objects*

then S preserves objects which are both admissible and cocomplete.

Theorem 7 *1.* $\mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$ *is the small globular colimit completion of* 1*.*

2. $\mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$ *is cartesian closed as a globular category.*

In fact the above results are true when $\mathbb{G}$ is replaced by an arbitrary small category $\mathbb{C}$.