Operads and polynomial 2-monads

Mark Weber

Creswick Victoria, June 2016

Some references

- (Gambino-Kock) *Polynomial functors and polynomial monads*, Math. Proc. Camb. Phil. Soc., 2013.
- (Batanin-Berger) *Homotopy theory for algebras over polynomial monads*, ArXiv:1305.0086.
- (W) *Polynomials in categories with pullbacks*.
- (W) *Operads as polynomial 2-monads*.

My papers plus these slides can be found at:

- https://sites.google.com/site/markwebersmaths/

Notation & terminology

The 2-category **Opd** has

- objects – coloured symmetric operads. So $T \in \mathbf{Opd}$ has a set of colours I, sets of operations $T(i_1, ..., i_n; j)$ for $i_k, j \in I$, satisfying the usual axioms.

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- $\phi : f \Rightarrow f'$ – unary operations $\phi_i \in T'(fi; f'i)$ satisfying:

$$\phi_j f(\alpha) = f'(\alpha)(\phi_{i_1}, ..., \phi_{i_n}) \qquad \forall\ \alpha \in T(i_1, ..., i_n; j)$$

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For $\mathcal{V}$ symmetric monoidal, the operad $\mathrm{End}(\mathcal{V})$ has colours the objects of $\mathcal{V}$ and

$$\mathrm{End}(\mathcal{V})(X_1, ..., X_n; Y) := \mathcal{V}(X_1 \otimes ... \otimes X_n, Y).$$

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The category of algebras of an operad T in $\mathcal{V}$ is

$$T\text{-}\mathrm{Alg}(\mathcal{V}) := \mathbf{Opd}(T, \mathrm{End}(\mathcal{V})).$$

Σ-free operads

(I, T) is Σ-**free** when the Σ_n-actions admit no fixed points, ie

$$\forall\ \alpha \text{ of arity } n \text{ and } \rho \in \Sigma_n,\ \ \alpha\rho = \alpha \Rightarrow \rho = 1_n$$

and then the induced endofunctor $\overline{T}$ on $\mathbf{Set}/I$ simplifies

$$\overline{T}(X_j)_{j\in I} = \coprod_{n\in\mathbb{N}} \left(\coprod_{\alpha:i_1,\ldots,i_n\to j} \prod_{k=1}^{n} X_{i_k} \right) \Big/ \Sigma_n = \coprod_{\alpha\in B} \prod_{k=1}^{n} X_{i_k}$$

where

$$B = \{\text{opn's of } T\} \Big/ \Sigma_n\text{-actions}$$

Definition of polynomial functor

Recall, for $f : I \to I'$ in a category $\mathcal{E}$ with pullbacks, one has

$$\begin{tikzcd}
\mathcal{E}/I \arrow[r, shift left=4ex, "\Sigma_f"] \arrow[r, phantom, shift left=2ex, "\perp"] \arrow[r, phantom, shift right=2ex, "\perp"] \arrow[r, dotted, shift right=4ex, "\Pi_f\ (\exists \text{ iff } f \text{ exponentiable})"'] & \mathcal{E}/I' \arrow[l, "\Delta_f" description]
\end{tikzcd}$$

Definition of polynomial functor

Recall, for $f : I \to I'$ in a category $\mathcal{E}$ with pullbacks, one has

$$\begin{tikzcd}
\mathcal{E}/I \arrow[rr, shift left=3ex, "\Sigma_f"] \arrow[rr, phantom, shift left=1.5ex, "\perp"] \arrow[rr, phantom, shift right=1.5ex, "\perp"] \arrow[rr, dotted, shift right=3ex, "\Pi_f\ (\exists \text{ iff } f \text{ exponentiable})"'] && \mathcal{E}/I' \arrow[ll, "\Delta_f" description]
\end{tikzcd}$$

Functor	General process	Process when $\mathcal{E} = \mathbf{Set}$
Σ_f	compose with f	sum along f's fibres
Δ_f	pb along f	duplicate along f's fibres
Π_f	dpb along f	product along f's fibres

Definition of polynomial functor

Recall, for $f : I \to I'$ in a category $\mathcal{E}$ with pullbacks, one has

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\mathcal{E}/I \arrow[r, shift left=8, "\Sigma_f"] \arrow[r, phantom, shift left=4, "\perp"] & \mathcal{E}/I' \arrow[l, "\Delta_f" description] \arrow[l, phantom, shift left=4, "\perp"] \\
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E}/I \arrow[r, dotted, "\Pi_f\ (\exists \text{ iff } f \text{ exponentiable})"'] & \mathcal{E}/I'
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A **polynomial functor** over $\mathcal{E}$ is a composite of such functors.

Polynomials

A **polynomial** from I to J in a category $\mathcal{E}$ with pullbacks is a diagram

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & J \end{tikzcd}$$

in which p is exponentiable.

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For a Σ-free operad (I, T), B as defined above and

$$E = \{\text{opn's of } T \text{ with chosen input}\} \Big/ \Sigma_n\text{-actions}$$

fit into a polynomial

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & I \end{tikzcd}$$

and its **associated polynomial functor** $\Sigma_t \Pi_p \Delta_s$ is $\overline{T}$.

The bicategory of polynomials

For $\mathcal{E}$ with pullbacks, one has a bicategory $\mathbf{Poly}_{\mathcal{E}}$ whose objects are those of $\mathcal{E}$, an arrow I to J is a polynomial as above, and a 2-cell from p to q is a diagram

$$\begin{tikzcd}
 & E \arrow[dl, "p_1"'] \arrow[r, "p_2"] \arrow[dd, "f_0"'] \arrow[ddr, phantom, "pb"] & B \arrow[dd, "f_1"] \arrow[dr, "p_3"] & \\
I & & & J \\
 & E' \arrow[ul, "q_1"] \arrow[r, "q_2"'] & B' \arrow[ur, "q_3"'] &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & E \arrow[ld, "p_1"'] \arrow[r, "p_2"] \arrow[dd, "f_0"'] \arrow[rdd, phantom, "pb"] & B \arrow[rd, "p_3"] \arrow[dd, "f_1"] & \\
I & & & J \\
 & E' \arrow[lu, "q_1"] \arrow[r, "q_2"'] & B' \arrow[ru, "q_3"'] &
\end{tikzcd}$$

The process of taking the corresponding polynomial functor is the effect on 1-cells of a homomorphism of bicategories

$$\mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \longrightarrow \mathbf{CAT} \qquad I \mapsto \mathcal{E}/I$$

Composition (= substitution) of polynomials

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "s_1"'] \arrow[r, "p_1"'] & \bullet \arrow[r, "t_1"'] & \bullet & \bullet \arrow[l, "s_2"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "t_2"'] & \bullet
\end{tikzcd}$$

Composition (= substitution) of polynomials

$$\begin{tikzcd}
& & & \bullet \arrow[dl] \arrow[dr] \arrow[d, phantom, "pb"] & & & \\
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Composition (= substitution) of polynomials

At this point one can consider the category of triples of morphisms (α, β, γ) as shown

$$\begin{tikzcd}
 & & & \bullet \arrow[rr, dotted, "\beta"'] \arrow[d, dotted, "\alpha"'] \arrow[drr, phantom, "pb"] & & \bullet \arrow[dd, dotted, "\gamma"] & \\
 & & & \bullet \arrow[dl] \arrow[dr, "f"] \arrow[d, phantom, "pb"] & & & \\
\bullet & \bullet \arrow[l, "s_1"'] \arrow[r, "p_1"'] & \bullet \arrow[r, "t_1"'] & \bullet & \bullet \arrow[l, "s_2"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "t_2"'] & \bullet
\end{tikzcd}$$

making the square with boundary $(f\alpha, p_2, \gamma, \beta)$ a pullback.

Composition (= substitution) of polynomials

The terminal such is the **distributivity pullback** of f along p_2.

$$\begin{tikzcd}
 & & & \bullet \arrow[rr, "q"] \arrow[d, "p"'] \arrow[dddrr, phantom, "\color{red}{dpb}"] & & \bullet \arrow[dd, "r"] & \\
 & & & \bullet \arrow[dl] \arrow[dr, "f"] \arrow[d, phantom, "pb"] & & & \\
\bullet & \bullet \arrow[l, "s_1"] \arrow[r, "p_1"'] & \bullet \arrow[r, "t_1"'] & \bullet & \bullet \arrow[l, "s_2"] \arrow[r, "p_2"'] & \bullet \arrow[r, "t_2"'] & \bullet
\end{tikzcd}$$

$$\Pi_{p_2}\Sigma_f \cong \Sigma_r\Pi_q\Delta_p \qquad r = \Pi_{p_2}(f)$$

Composition (= substitution) of polynomials

$$\begin{tikzcd}
& & & \bullet \arrow[rrr] \arrow[ddl] & & & \bullet \arrow[rrr] \arrow[d] & & & \bullet \arrow[ddr] & & & \\
& & & & pb & & \bullet \arrow[dll] \arrow[drr] \arrow[d, phantom, "pb"] & & dpb & & & & \\
\bullet & & \bullet \arrow[ll, "s_1"'] \arrow[rr, "p_1"'] & & \bullet \arrow[rr, "t_1"'] & & \bullet & & \bullet \arrow[ll, "s_2"'] \arrow[rr, "p_2"'] & & \bullet \arrow[rr, "t_2"'] & & \bullet
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Composition (= substitution) of polynomials

$$\begin{tikzcd}
 & & & \bullet \arrow[rrr] \arrow[rrrrrr, bend left, "p_3"] \arrow[ddlll, "s_3"'] \arrow[ddl] \arrow[ddr, phantom, "pb"] & & & \bullet \arrow[rrr] \arrow[d] & & & \bullet \arrow[ddr] \arrow[ddrrr, "t_3"] \arrow[ddl, phantom, "dpb"] & & & \\
 & & & & & & \bullet \arrow[dll] \arrow[drr] \arrow[d, phantom, "pb"] & & & & & & \\
\bullet & & \bullet \arrow[ll, "s_1"] \arrow[rr, "p_1"'] & & \bullet \arrow[rr, "t_1"'] & & \bullet & & \bullet \arrow[ll, "s_2"] \arrow[rr, "p_2"'] & & \bullet \arrow[rr, "t_2"'] & & \bullet
\end{tikzcd}$$

When p_1 and p_2 are identities, this reduces to the usual pullback-composition of spans.

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$$\begin{tikzcd}
& I \arrow[r, "1"] \arrow[dl, "1"'] \arrow[dd, dotted] \arrow[ddr, phantom, "pb"] & I \arrow[dr, "1"] \arrow[dd, dotted] & \\
I & & & I \\
& E \arrow[ul, "s"] \arrow[r, "p"'] & B \arrow[ur, "t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& E' \arrow[r, "p'"] \arrow[dl, "s'"'] \arrow[dd, dotted] \arrow[ddr, phantom, "pb"] & B' \arrow[dr, "t'"] \arrow[dd, dotted] & \\
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A Σ-free operad T determines a polynomial monad $\mathcal{N}_0 T$ over **Set**.

Morphisms of polynomial monads

The morphisms of monads relevant for us involve some base-change. To this end note that for any $f : I \to I'$ in $\mathcal{E}$ one has

$$f^{\bullet} \; : \; I \xleftarrow{1} I \xrightarrow{1} I \xrightarrow{f} I' \quad \dashv \quad f_{\bullet} \; : \; I' \xleftarrow{f} I \xrightarrow{1} I \xrightarrow{1} I$$

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and so one can define a functor $\text{PMnd}_{\mathcal{E}} : \mathcal{E}^{\text{op}} \to \mathbf{Cat}$ by

$$I \mapsto \text{Mon}\left(\mathbf{Poly}_{\mathcal{E}}(I, I)\right) \qquad f \mapsto f_{\bullet} \circ (-) \circ f^{\bullet}$$

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Definition : the category of polynomial monads

$$\mathbf{PolyMnd}_{\mathcal{E}} = \int \mathrm{PMnd}_{\mathcal{E}}$$

Morphisms of polynomial monads

In $\mathcal{E}$ the data of a morphism of $\mathbf{PolyMnd}_{\mathcal{E}}$ is

$$\begin{tikzcd}
I \arrow[d, "f"'] & E \arrow[l, "s"'] \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[r, "t"] \arrow[d] & I \arrow[d, "f"] \\
I' & E' \arrow[l, "s'"] \arrow[r, "p'"'] & B' \arrow[r, "t'"'] & I'
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\end{tikzcd}$$

In $\mathbf{Poly}_{\mathcal{E}}$ this is an **adjunction of monads** $(I, \mathbf{p}) \to (I', \mathbf{p}')$

$$\begin{tikzcd}
I \arrow[loop left, "\mathbf{p}"] \arrow[r, shift left, "f^\bullet"] \arrow[r, phantom, "\perp"] & I' \arrow[l, shift left, "f_\bullet"] \arrow[loop right, "\mathbf{p}'"]
\end{tikzcd}$$

$$\left.\begin{array}{c}
f^\bullet \mathbf{p} f_\bullet \Longrightarrow \mathbf{p}' \\ \hline
\mathbf{p} \Longrightarrow f_\bullet \mathbf{p}' f^\bullet \\ \hline
f^\bullet \mathbf{p} \Longrightarrow \mathbf{p}' f^\bullet \\ \hline
\mathbf{p} f_\bullet \Longrightarrow f_\bullet \mathbf{p}'
\end{array}\right| \leftarrow \begin{array}{c}\text{equiv.data}\\ \text{by}\\ f^\bullet \dashv f_\bullet\end{array}$$

Notation for adjunctions of monads

Adjunctions of monads make sense in any bicategory, and in particular in **CAT** an adjunction $F : (\mathcal{E}, T) \to (\mathcal{F}, S)$ consists of

$$\begin{tikzcd} \mathcal{E} \arrow[loop left, "T"] \arrow[r, shift left=2, "F_!"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, shift left=2, "F^*"] \arrow[loop right, "S"] \end{tikzcd}$$

$$\frac{F^c \ : \ F_! T \Rightarrow SF_!}{F^l \ : \ TF^* \Rightarrow F^* S}$$

in which F^l (resp. F^c) endows F^* (resp. $F_!$) with the structure of a lax (resp. colax) monad morphism. To give F^l is to give a lifting of F^* to S-Alg $\to$ T-Alg.

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Apply $\mathbf{P}_{\mathcal{E}}$ to the previous slide gives an adjunction of monads where

$$F_! = \Sigma_f \qquad\qquad F^* = \Delta_f .$$

Theorem

(J. Kock, Szawiel-Zawadowski)

1. $T \mapsto \mathcal{N}_0 T$ *gives an equivalence between the category of* Σ*-free operads and the full subcategory of* $\mathbf{PolyMnd}_{\mathbf{Set}}$ *consisting of those polynomial monads*

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & I \end{tikzcd}$$

in which p *has finite fibres.*

2. *For all* T, $\overline{T}\text{-Alg} \cong T\text{-Alg}(\mathbf{Set})$ *where* $\overline{T} = \mathbf{P}_{\mathbf{Set}}(\mathcal{N}_0 T)$.

Polynomials in 2-categories

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$$\mathbf{Poly}_{\mathcal{E}} \longrightarrow \mathbf{2\text{-}CAT}$$

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Important case for us: $\mathcal{E} = \mathbf{Cat}$.

Examples

Denoting $\mathbb{P}$ the permutation category and $\mathbb{P}_*$ the based version, one has a polynomial

$$\begin{tikzcd} 1 & \mathbb{P}_* \arrow[l] \arrow[r, "U^{\mathbb{P}}"] & \mathbb{P} \arrow[r] & 1 \end{tikzcd}$$

which underlies a polynomial monad, and the corresponding 2-monad is denoted $\mathbf{S}$.

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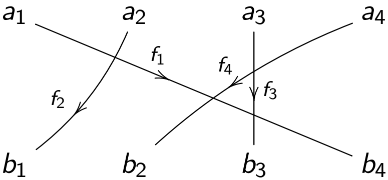


Examples

Noting $\mathbb{N} = \mathrm{ob}(\mathbb{P})$ and denoting $\mathbb{N}_* = \mathrm{ob}(\mathbb{P}_*)$ one has a polynomial

$$\begin{tikzcd}1 & \mathbb{N}_* \arrow[l] \arrow[r, "U^{\mathbb{N}}"] & \mathbb{N} \arrow[r] & 1\end{tikzcd}$$

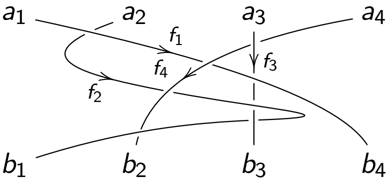
which underlies a polynomial monad over **Set**. Regarding this as a (componentwise discrete) polynomial monad in **Cat**, the corresponding 2-monad **M** is the sub-2-monad of **S** determined by the levelwise maps.

Examples

Denoting $\mathbb{B}$ the braid category and $\mathbb{B}_*$ the based version, one has a polynomial

$$\begin{tikzcd} 1 & \mathbb{B}_* \arrow[l] \arrow[r, "U^{\mathbb{B}}"] & \mathbb{B} \arrow[r] & 1 \end{tikzcd}$$

which underlies a polynomial monad, and the corresponding 2-monad is denoted $\mathbf{B}$. For a category A, objects of $\mathbf{B}A$ are finite sequences of objects of A, and morphisms are braids labelled by morphisms of A as in



2-monad algebras and algebra morphisms

	axiom holds up to	**S**-algebra	**S**-morphism
strict	equality	symmetric strict monoidal category	symmetric strict monoidal functor
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2-monad algebras and algebra morphisms

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Internal structure via (co)lax algebra morphisms

$\mathcal{V}$: symmetric monoidal category = pseudo **S**-algebra.
Lax **S**-morphism $1 \to \mathcal{V}$ is a commutative monoid in $\mathcal{V}$.
Colax **S**-morphism $1 \to \mathcal{V}$ is a cocommutative comonoid in $\mathcal{V}$.

2-monad algebras and algebra morphisms

	axiom holds up to	**M**-algebra	**M**-morphism
strict	equality	strict monoidal category	strict monoidal functor
pseudo	coherent isomorphism	monoidal category	(strong) monoidal functor
lax	coherent 2-cell	lax monoidal category	lax monoidal functor

Internal structure via (co)lax algebra morphisms

$\mathcal{V}$: monoidal category = pseudo **M**-algebra.
Lax **M**-morphism $1 \to \mathcal{V}$ is a monoid in $\mathcal{V}$.
Colax **M**-morphism $1 \to \mathcal{V}$ is a comonoid in $\mathcal{V}$.

2-monad algebras and algebra morphisms

	axiom holds up to	**B**-algebra	**B**-morphism
strict	equality	braided strict monoidal category	braided strict monoidal functor
pseudo	coherent isomorphism	braided monoidal category	braided (strong) monoidal functor
(co)lax	coherent 2-cell	braided functor (co)operad	braided (co)lax monoidal functor

Internal structure via (co)lax algebra morphisms

$\mathcal{V}$: braided monoidal category = pseudo **B**-algebra.
Lax **B**-morphism $1 \to \mathcal{V}$ is a commutative monoid in $\mathcal{V}$.
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Internal algebras

Definition

Given an adjunction $F : (\mathcal{K}, T) \longrightarrow (\mathcal{L}, S)$ of 2-monads and an S-algebra A, the category T-$\mathrm{Alg}(A)$ of **algebras of T internal to A** is the category of lax morphisms $1 \to F^*A$ and algebra 2-cells between them.

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Examples

When F is the identity on $\mathbf{S}$, $\mathbf{B}$ or $\mathbf{M}$ one recovers the category of commutative monoids in a symmetric or braided monoidal category, and that of monoids in a monoidal category.

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Examples

When $F_! = F^* = 1_{\mathbf{Cat}}$ and $F^l = F^c$ is the inclusion $\mathbf{M} \hookrightarrow \mathbf{S}$ (resp. $\mathbf{M} \hookrightarrow \mathbf{B}$) one recovers the category of monoids in a symmetric (resp. braided) monoidal category.

Constructing a polynomial 2-monad from an operad

Let T be an operad with set of colours I. There's a functor

$$\mathrm{Op}_T : \mathbb{P}^{\mathrm{op}} \longrightarrow \mathbf{Set} \qquad\qquad n \mapsto \{\text{n-ary opn's of } T\}$$

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$$B_T = \int \mathrm{Op}_T \qquad E_T = \int \mathrm{Op}_T(U^{\mathbb{P}})^{\mathrm{op}}$$

giving an adjunction $\mathcal{N}T \to \mathbf{S}$ of polynomial monads:

$$\begin{tikzcd}
I \arrow[d] & E_T \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & B_T \arrow[r] \arrow[d, "\text{dfib}"] & I \arrow[d] \\
1 & \mathbb{P}_* \arrow[l] \arrow[r] & \mathbb{P} \arrow[r] & 1
\end{tikzcd}$$

Theorem

1. *$T \mapsto \mathcal{N}T \to \mathbf{S}$ gives an equivalence between the category* **Opd** *and the full subcategory of* $\mathbf{PolyMnd}_{\mathbf{Cat}}/\mathbf{S}$ *consisting of those*

$$\begin{tikzcd}
I \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & B \arrow[r] \arrow[d, "d"] & I \arrow[d] \\
1 & \mathbb{P}_* \arrow[l] \arrow[r] & \mathbb{P} \arrow[r] & 1
\end{tikzcd}$$

such that d is a discrete fibration.

2. *For any symmetric monoidal category $\mathcal{V}$ and operad T,* $\widetilde{T}\text{-Alg}(\mathcal{V}) \cong T\text{-Alg}(\mathcal{V})$ *where* $\widetilde{T} = \mathbf{P}_{\mathbf{Cat}}(\mathcal{N}(T))$.

Variants of this result

1. The full subcategory of $\mathbf{PolyMnd}_{\mathbf{Cat}}/\mathbf{S}$ determined by $I = 1$ (no condition on d) is the category of Kelly's clubs. Thus $\mathbf{PolyMnd}_{\mathbf{Cat}}/\mathbf{S}$ contains clubs and operads as full subcategories.
2. In a similar way non-Σ-operads are identified within $\mathbf{PolyMnd}_{\mathbf{Cat}}/\mathbf{M}$, and braided operads are identified within $\mathbf{PolyMnd}_{\mathbf{Cat}}/\mathbf{B}$.
3. An adjunction of polynomial monads into $\mathbf{S}$ together with the structure of a split fibration on d, is the same as giving a **Cat**-operad.

T-algebras vs $\widetilde{T}$-algebras

Strict $\widetilde{\mathrm{Com}}$-algebras are symmetric strict monoidal categories whereas algebras of Com in **Cat** are symmetric strict monoidal categories whose symmetries are identities.

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An $\widetilde{Ass}$-algebra structure on a category amounts to a tensor product functor $\otimes_\rho : \mathcal{V}^n \to \mathcal{V}$ for each $\rho \in \Sigma_n$, together with isomorphisms $\otimes_{\rho_1} c_{\rho_2} \cong \otimes_{\rho_1 \rho_2}$ (where $c_{\rho_2} : \mathcal{V}^n \to \mathcal{V}^n$ permutes the factors according to ρ_2), and the usual coherences for monoidal categories adapted to this situation.

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In general: $\widetilde{T}$-algebras are **weakly equivariant** morphisms of operads $T \to \text{End}(\mathbf{Cat})$.

$\overline{T}$ vs $\widetilde{T}$

For any operad (I, T), applying $\mathbf{P}_{\mathbf{Cat}}$ to

$$\begin{tikzcd}
 & E_T^{[1]} \arrow[dl, "s_T^{[1]}"'] \arrow[r, "p_T^{[1]}"] \arrow[dd, bend right, "d_{E_T}"'] \arrow[dd, bend left, "c_{E_T}"] \arrow[dd, phantom, "\overset{\alpha_{E_T}}{\Rrightarrow}"] & B_T^{[1]} \arrow[dr, "t_T^{[1]}"] \arrow[dd, bend right, "d_{B_T}"'] \arrow[dd, bend left, "c_{E_T}"] \arrow[dd, phantom, "\overset{\alpha_{B_T}}{\Rrightarrow}"] & \\
I & & & I \\
 & E_T \arrow[ul, "s_T"] \arrow[r, "p_T"'] & B_T \arrow[ur, "t_T"'] &
\end{tikzcd}$$

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I & & & I \\
& E_T \arrow[ul, "s_T"] \arrow[r, "p_T"'] & B_T \arrow[ur, "t_T"'] &
\end{tikzcd}$$

gives a 2-cell α_T of 2-monads whose coidentifier q_T

$$\begin{tikzcd}
\widetilde{T}^{[1]}_{\Sigma} \arrow[r, shift left=2, "d_T"] \arrow[r, shift right=2, "c_T"'] \arrow[r, phantom, "\Downarrow\alpha_T"] & \widetilde{T} \arrow[r, "q_T"] & \overline{T}
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\end{tikzcd}$$

is constructed as in $\mathrm{End}(\mathbf{Cat}/I)$, $\overline{T}$-algebras are algebras of T in **Cat**, and q_T induces the inclusion of operad morphisms $T \to \mathrm{End}(\mathbf{Cat})$ amongst the weakly-equivariant ones.

$\overline{T}$ in the Σ-free case

When T is Σ-free, q_T can be obtained by applying $\mathbf{P}_{\mathbf{Cat}}$ to

$$\begin{tikzcd}
 & E_T^{[1]} \arrow[r, "p_T^{[1]}"] \arrow[dl, "s_T^{[1]}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{\alpha_{E_T}}{\Rightarrow}"] & B_T^{[1]} \arrow[dr, "t_T^{[1]}"] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{\alpha_{B_T}}{\Rightarrow}"] & \\
I & E_T \arrow[l, "s_T"'] \arrow[r, "p_T"'] \arrow[d, "q_{E_T}"] & B_T \arrow[r, "t_T"'] \arrow[d, "q_{B_T}"] & I \\
 & \pi_0 E_T \arrow[ul, "\pi_0 s_T"] \arrow[r, "\pi_0 p_T"'] & \pi_0 B_T \arrow[ur, "\pi_0 t_T"'] &
\end{tikzcd}$$

which is a coidentifier in $\mathbf{Poly}_{\mathbf{Cat}}(I, I)$. The bottom polynomial monad is the Kock/Szabiel-Zawadowski polynomial for T.

Review of Lack model structures

Cat has a model structure in which the equivalences are equivalences of categories, and fibrations are functors with the isomorphism lifting property (the "isofibrations"). This is known as the "folk", "natural" or "categorical" model structure on **Cat**.

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Theorem

If T is Σ-free then $q_T : \widetilde{T} \to \overline{T}$ induces a

1. *Quillen equivalence between* $\overline{T}$-Alg_s *and* $\widetilde{T}$-Alg_s *with respect to the Lack model structures.*
2. *biequivalence between* $\overline{T}$-Alg *and* $\widetilde{T}$-Alg.
3. *biequivalence between* Ps-$\overline{T}$-Alg *and* Ps-$\widetilde{T}$-Alg.

Operads and polynomial 2-monads II

Mark Weber

Creswick Victoria, June 2016

Some references

- (Gambino-Kock) *Polynomial functors and polynomial monads*, Math. Proc. Camb. Phil. Soc., 2013.
- (Batanin-Berger) *Homotopy theory for algebras over polynomial monads*, ArXiv:1305.0086.
- (W) *Polynomials in categories with pullbacks*.
- (W) *Operads as polynomial 2-monads*.
- (W) *Internal algebra classifiers as codescent objects of crossed internal categories*.
- (W) *Algebraic Kan extensions along morphisms of internal algebra classifiers*.

My papers plus the slides for both talks can be found at:

- https://sites.google.com/site/markwebersmaths/

A **polynomial** from I to J in a category (2-category) $\mathcal{E}$ with pullbacks is a diagram

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & J \end{tikzcd}$$

in which p is exponentiable. The **associated polynomial functor** $\mathcal{E}/I \to \mathcal{E}/J$ is $\Sigma_t \Pi_p \Delta_s$.

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Polynomials in $\mathcal{E}$ form a bicategory (2-bicategory) $\mathbf{Poly}_{\mathcal{E}}$ and taking the associated polynomial functor is the effect on 1-cells of a homomorphism

$$\mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \longrightarrow \mathbf{CAT} \qquad \mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \longrightarrow \mathbf{2\text{-}CAT}$$

Today we shall focus on polynomial monads in **Cat**

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & I \end{tikzcd}$$

in which I is discrete, E and B are groupoids, and p is a discrete fibration with finite fibres. Let's call these **operadic polynomial monads**.

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in which I is discrete, E and B are groupoids, and p is a discrete fibration with finite fibres. Let's call these **operadic polynomial monads**. Applying $\mathbf{P}_{\mathbf{Cat}}$ to a morphism of such

$$\begin{tikzcd} I \arrow[d, "f"'] & E \arrow[l, "s"'] \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[r, "t"] \arrow[d] & I \arrow[d, "f"] \\ I' & E' \arrow[l, "s'"] \arrow[r, "p'"'] & B' \arrow[r, "t'"'] & I' \end{tikzcd}$$

produces an adjunction of 2-monads.

An **adjunction of 2-monads** $F : (\mathcal{K}, T) \to (\mathcal{L}, S)$ consists of

1. A 2-category $\mathcal{K}$ and a 2-monad T on $\mathcal{K}$.
2. A 2-category $\mathcal{L}$ and a 2-monad S on $\mathcal{L}$.
3. An adjunction $F_! \dashv F^* : \mathcal{L} \to \mathcal{K}$.
4. A lifting of $F^* : \mathcal{L} \to \mathcal{K}$ to the 2-categories of algebras S and T. This is equivalent to giving compatible 2-natural transformations

$$F^c : F_! T \Rightarrow SF_! \qquad\qquad F^l : TF^* \Rightarrow F^* S$$

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In such a formal set up one defines a **T-algebra internal to an S-algebra** A to be a lax morphism $1 \to F^* A$ of T-algebras (given a terminal object $1 \in \mathcal{K}$).

An operad T with set of colours I determines

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\end{tikzcd}$$

and thus an adjunction of 2-monads

$$\mathrm{Ar}_T : (\mathbf{Cat}/I, \widetilde{T}) \longrightarrow (\mathbf{Cat}, \mathbf{S})$$

an algebra of T in a symmetric monoidal category $\mathcal{V}$ in the usual sense, corresponds to a $\widetilde{T}$-algebra internal to $\mathcal{V}$ (viewed as a pseudo **S**-algebra).

Goals

For an adjunction $F : (\mathcal{K}, T) \to (\mathcal{L}, S)$ of 2-monads coming from a morphism of operadic polynomial monads, want to explain:

1. How to define S^T – the **universal** S**-algebra containing an internal** T**-algebra**.

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1. How to define S^T – the **universal S-algebra containing an internal T-algebra**.
2. How to obtain an explicit description of S^T from the data of F.
3. How to use these "internal algebra classifiers" to give explicit descriptions of various free constructions as left Kan extensions.

Revisiting Δ_+

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Composition with u induces

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Denoting $\mathbf{M}$ = 2-monad on **Cat** for monoidal categories,

$$\mathrm{Nerve}(\Delta_+) \;=\; \begin{tikzcd} \cdots & \mathbf{M}^3 1 \arrow[r, shift left=3, "\mu_{\mathbf{M}1}"] \arrow[r, "\mathbf{M}\mu_1" description] \arrow[r, shift right=3, "\mathbf{M}^2(!)"'] & \mathbf{M}^2 1 \arrow[r, shift left=3, "\mu_1"] \arrow[r, shift right=3, "\mathbf{M}(!)"'] & \mathbf{M}1 \arrow[l, "\mathbf{M}\eta_1" description] \end{tikzcd}$$

Definition

Given $F : (\mathcal{K}, T) \to (\mathcal{L}, S)$ the **internal T-algebra classifier** consists of a strict S-algebra S^T and an internal T-algebra $u : 1 \to F^*(S^T)$, such that for all strict S-algebras X, the functor

$$S\text{-}\mathrm{Alg}_{\mathrm{s}}(S^T, X) \longrightarrow T\text{-}\mathrm{Alg}_{\mathrm{l}}(1, F^* X)$$

given by precomposition with u is an isomorphism of categories.

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In the case $(\mathcal{L}, S) = (\mathcal{K}, T) = (\mathbf{Cat}, \mathbf{M})$ and F is the identity, we have

$$\mathbf{M}^{\mathbf{M}} = \Delta_+$$

Proposition

If F comes from a morphism of operadic polynomial monads, then S^T exists and composition with $u : 1 \to F^*(S^T)$ gives pseudonatural equivalences

$$\text{Ps-}S\text{-Alg}(S^T, X) \simeq \text{Ps-}T\text{-Alg}_{\mathrm{l}}(1, F^*X)$$

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Proof-ideas

Operadicity implies S is finitary, and so $S\text{-Alg}_{\text{s}}$ is lfp. The 2-functor $X \mapsto T\text{-Alg}_{\text{l}}(1, F^*X)$ is limit-preserving and thus representable.

Proposition

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Regarding $X \mapsto F^*X$ as the effect on objects of a 2-functor

$$S\text{-Alg}_{\mathsf{s}} \longrightarrow T\text{-Alg}_{\mathsf{l}},$$

S^T is the effect on 1 of its left adjoint.

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> Lack, *Codescent objects and coherence*.

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Adapting this to our more general situation gives the formula

$$S^T = \mathrm{CoDesc}\left(\begin{tikzcd} \cdots \; SF_!T^21 \arrow[r, shift left=2, "\mu^S_{F_!T1}S(F^c_{T1})"] \arrow[r, "SF_!\mu^T_1" description] \arrow[r, shift right=2, "SF_!T(!)"'] & SF_!T1 \arrow[r, shift left=2, "\mu^S_{F_!X}S(F^c_1)"] \arrow[r, shift right=2, "SF_!(!)"'] & SF_!1 \arrow[l, "SF_!\eta^T_1" description] \end{tikzcd}\right)$$

Definition of codescent object

If $X : \Delta^{\mathrm{op}} \to \mathcal{K}$ is a simplicial object in a 2-category $\mathcal{K}$ and $Y \in \mathcal{K}$, then a **codescent cocone** for X with vertex Y consists of (q_0, q_1)

$$\begin{tikzcd}
\cdots & X_2 \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_0"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, "s_0" description] \arrow[r, "q_0", "q_1 : q_0 d_1 \Rightarrow q_0 d_0"'] & Y
\end{tikzcd}$$

such that

$$q_1 s_0 = \mathrm{id} \qquad (q_1 d_0)(q_1 d_2) = (q_1 d_1)$$

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& & & \arrow[r, phantom, "q_1 : q_0 d_1 \Rightarrow q_0 d_0"] &
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such that

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The universal such is denoted $Y = \mathrm{CoDesc}(X)$ and is called the **codescent object** of X.

Example

Every category is the codescent object of its nerve, regarded as a componentwise-discrete category object in **Cat**, and q_0 in this case is the inclusion of objects.

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Regarding a 2-category X as a category object in **Cat**

$$\begin{tikzcd}
\cdots & X_2 \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_0"'] & X_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & X_0 \arrow[l, "s_0" description] & (X_0 \text{ discrete})
\end{tikzcd}$$

one has $\mathrm{CoDesc}(X) = \pi_{0*}X$.

A **double category** is a category object $X : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ in $\mathbf{Cat}$. We regard the arrows of X_0 as **vertical arrows**, and the objects of X_1 as **horizontal arrows**.

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Definition

A **crossed double category** is a double category X together with the structure of a split opfibration on $d_0 : X_1 \to X_0$ such that

$$\begin{tikzcd}
X_0 \arrow[r, "s_0"] \arrow[dr, "1"'] & X_1 \arrow[d, "d_0"] & X_2 \arrow[l, "d_1"'] \arrow[dl, "d_0^2"] \\
& X_0 &
\end{tikzcd}$$

are morphisms of split opfibrations over X_0.

To give a cleavage for $d_0 : X_1 \to X_0$ is to give, for all f and g, distinguished squares

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\kappa"] & x \arrow[d, "g"] \\ z \arrow[r, "r"'] & y \end{tikzcd}$$

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z \arrow[r, "r"'] & y
\end{tikzcd}$$

One has a 2-category $\mathrm{Cnr}(X)$ with objects those of X, arrows, horizontal composition and 2-cells as in:

$$\begin{tikzcd}
x \arrow[d, "g"'] & \\
a \arrow[r, "f"'] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[d, "g_1"] \arrow[dd, bend right=50, "g_3"'] & & \\
a \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\kappa"] & y \arrow[d, "g_2"] & \\
c \arrow[r] \arrow[rr, bend right, "f_3"'] & b \arrow[r, "f_2"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[d, "g_1"] \arrow[dd, bend right=50, "g_2"'] & \\
a \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\alpha"] & y \arrow[d, equal] \\
b \arrow[r, "f_2"'] & y
\end{tikzcd}$$

Examples

When the horizontal category $= \Delta$, vertical category is a groupoid, and $d_0 : X_1 \to X_0$ a **discrete** opfibration then $\mathrm{Cnr}(X)$ is a crossed simplicial group à la Loday-Fiedorowicz.

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When the horizontal category $= \Delta$, vertical category is a groupoid, and $d_0 : X_1 \to X_0$ a **discrete** opfibration then $\mathrm{Cnr}(X)$ is a crossed simplicial group à la Loday-Fiedorowicz.

Examples

When $F : (\mathbf{Cat}/I, T) \to (\mathbf{Cat}/I', S)$ comes from a morphism of operadic polynomial monads, then

$$\begin{tikzcd}
\cdots & SF_! T^2 1 \arrow[r, shift left=2, "\mu^S_{F_! T1} S(F^c_{T1})"] \arrow[r, "SF_! \mu^T_1" description] \arrow[r, shift right=2, "SF_! T(!)"'] & SF_! T1 \arrow[r, shift left=2, "\mu^S_{F_! X} S(F^c_1)"] \arrow[r, shift right=2, "SF_!(!)"'] & SF_! 1 \arrow[l, "SF_! \eta^T_1" description]
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is componentwise a crossed double category.

Theorem

For X a crossed double category

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This result works if you replace **Cat** by the 2-category $\mathrm{Cat}(\mathcal{E})$ of categories internal to $\mathcal{E}$, where $\mathcal{E}$ is a category with pullbacks and pullback-stable reflexive coequalisers.

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This result works if you replace **Cat** by the 2-category $\mathrm{Cat}(\mathcal{E})$ of categories internal to $\mathcal{E}$, where $\mathcal{E}$ is a category with pullbacks and pullback-stable reflexive coequalisers. The statement

$$\mathrm{CoDesc}(X) = \mathrm{CoDesc}(\mathrm{Cnr}(\mathrm{X}))$$

is even more general, just requiring pullbacks in $\mathcal{E}$.

Example

$\mathbf{S}^{\mathbf{S}}$, the free symmetric monoidal category containing a commutative monoid, is computed as

$$\pi_{0*}\mathrm{Cnr}\left(\begin{tikzcd}\cdots & \mathbf{S}^3 1 \arrow[r, shift left=2, "\mu_{\mathbf{S}1}"] \arrow[r, "\mathbf{S}\mu_1" description] \arrow[r, shift right=2, "\mathbf{S}^2(!)"'] & \mathbf{S}^2 1 \arrow[r, shift left=2, "\mu_1"] \arrow[r, shift right=2, "\mathbf{S}(!)"'] & \mathbf{S}1 \arrow[l, "\mathbf{S}\eta_1" description]\end{tikzcd}\right)$$

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The crossed double category has horizontal category $= \Delta_+$, vertical category $=$ the permutation category, and squares are squares that commute in $\mathbf{Set}_{\mathrm{fin}}$.

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The crossed double category has horizontal category $= \Delta_+$, vertical category $=$ the permutation category, and squares are squares that commute in $\mathbf{Set}_{\mathrm{fin}}$. The result $\mathbf{S}^{\mathbf{S}} = \mathbf{Set}_{\mathrm{fin}}$ and the crossed structure comes from the unique factorisation of any function $f : m \to n$ between finite ordinals as $f = \phi\rho$, where ρ is bijective, ϕ is order preserving, and ρ is order-preserving on the fibres of ρ.

Braided version of the previous example

Using our formula for $\mathbf{B}^{\mathbf{B}}$, and the work of Lavers (*The theory of vines*) one obtains the category of vines as the free braided monoidal category containing a commutative monoid.

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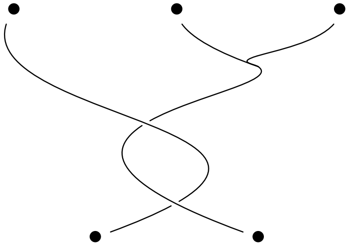

Symmetric monoidal categories from operads

For $F = \mathrm{Ar}_T : (\mathbf{Cat}/I, \widetilde{T}) \to (\mathbf{Cat}, \mathbf{S})$, $\mathbf{S}^{\widetilde{T}}$ has the universal properties

$$\mathbf{S}\text{-}\mathrm{Alg}_{\mathrm{s}}(\mathbf{S}^{\widetilde{T}}, \mathcal{V}) \cong T\text{-}\mathrm{Alg}(\mathcal{V}) \qquad \mathrm{Ps}\text{-}\mathbf{S}\text{-}\mathrm{Alg}(\mathbf{S}^{\widetilde{T}}, \mathcal{V}) \simeq T\text{-}\mathrm{Alg}(\mathcal{V})$$

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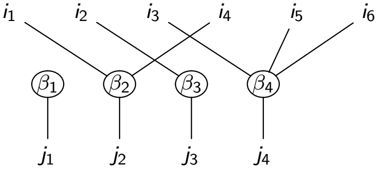
By the formula $\mathbf{S}^{\widetilde{T}}$ has as objects finite sequences of colours, and morphisms functions between indexing sets whose fibres are labelled by the operations of T. For example:



Braided Feynman categories?

Theorem (with M. Batanin and J. Kock)

A Feynman category (à la Kaufmann-Ward) is a category which is equivalent to $\mathbf{S}^{\widetilde{T}}$ for some operad T.

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For a braided operad T, the explicit description of $\mathbf{B}^{\widetilde{T}}$ is similar to $\mathbf{S}^{\widetilde{T}}$ except with indexing vines replacing indexing functions in the description of the morphisms.

Free properads

Let $\mathbb{C}$ be the groupoid such that symmetric bicoloured collections are functors out of $\mathbb{C}$. Let Prpd be the operad for properads. There's an operad morphism $\iota : \mathbb{C} \to \mathrm{Prpd}$.

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- A symmetric bicoloured collection in $\mathcal{V}$ "is" a symmetric strong monoidal functor $\mathbf{S}(\mathbb{C}) \to \mathcal{V}$.

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- The free properad on a symmetric bicoloured collection is computed by left Kan extending along

$$\mathbf{S}^{\widetilde{\iota}} : \mathbf{S}(\mathbb{C}) \longrightarrow \mathbf{S}^{\widetilde{\mathrm{Prpd}}}$$

Modular envelope

Let CycOp be the operad for cyclic operads, ModOp be the operad for modular operads and $\iota : \mathrm{CycOp} \to \mathrm{ModOp}$ be the inclusion.

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- A modular operad in $\mathcal{V}$ is a symmetric strong monoidal functor $\mathbf{S}^{\widetilde{ModOp}} \to \mathcal{V}$.
- The modular envelope of a cyclic operad is obtained by left Kan extending along

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Colimits in categories of algebras of an operad

Let $\mathbb{C}$ be a category, T be an operad and $\iota : \mathbb{C} \otimes T \to T$ be the "projection", where $\otimes$ is the Boardman-Vogt tensor product.

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Questions

Why do the left Kan extensions in such cases produce symmetric strong monoidal functors?

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Questions

Why do the left Kan extensions in such cases produce symmetric strong monoidal functors? More generally, when is the left Kan extension of a pseudo morphism also a pseudo morphism?

Given adjunctions of 2-monads

$$\begin{tikzcd}
(\mathcal{K}_1, T_1) \arrow[rr, "G"] \arrow[dr, "F_1"'] & & (\mathcal{K}_2, T_2) \arrow[dl, "F_2"] \\
& (\mathcal{L}, S) &
\end{tikzcd}$$

one regards S as the type of ambient structure, T_2 and T_1 as the types of internal structures. One has forgetful functors

$$U_A : T_2\text{-Alg}(A) \longrightarrow T_1\text{-Alg}(A)$$

for all S-algebras A.

Theorem

If all these 2-monads arise from operadic polynomial monads and A is algebraically cocomplete, then the left adjoint to U_A is computed by left Kan extending along the strict S-algebra morphism $\mathbf{S}^G : \mathbf{S}^{T_1} \to \mathbf{S}^{T_2}$.

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Algebraic cocompleteness says that pointwise left Kan extensions into in $\mathcal{L}$ into A are compatible with the action $SA \to A$. There is a general result (Melliès-Tabareau-Koudenburg) that pointwise left Kan extending a pseudo morphism into such an A, along an algebra morphism whose algebra square is exact in the sense of Guitart, is again a pseudo morphism.

To apply the Melliès-Tabareau-Koudenburg result we need to know that the algebra square for $\mathbf{S}^G$ is exact.

To apply the Melliès-Tabareau-Koudenburg result we need to know that the algebra square for $\mathbf{S}^G$ is exact. This follows from the following general result.

Theorem

Suppose that $\mathcal{S}$ is a pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

of crossed double categories in which g is an internal discrete fibration and f_0 is an opfibration. Then $\mathrm{CoDesc}(\mathcal{S})$ is exact.