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Mark Weber

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joint in part with Michael Batanin and Denis-Charles Cisinski

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Relevant Properties For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

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Relevant Properties For any category V one may consider the category \mathcal{GV} of graphs enriched in V.

objects : $\mathcal{GV} \to \mathbf{Set}$

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In particular

$$\mathcal{G}1 = \mathbf{Set}$$
 $\mathcal{G}\mathbf{Set} = \mathbf{Graph}$

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the category of *n*-globular sets is \mathcal{G}^n **Set**, and \mathcal{G} **Glob** \cong **Glob**.

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Batanin: higher categorical structures are algebras of monads on \mathcal{G}^n **Set** or **Glob**.

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Useful abstraction: globular higher category theory is about monads defined on \mathcal{GV} over **Set**.

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Batanin: higher categorical structures are algebras of monads on \mathcal{G}^n **Set** or **Glob**.

Useful abstraction: globular higher category theory is about monads defined on \mathcal{GV} over **Set**. The monads of interest come from distributive lax monoidal structures on V.

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Relevant Properties Let V be a category. We shall use the following notations

$$\mathop{\mathsf{E}}_{1\leq i\leq n} Z_i \qquad \mathop{\mathsf{E}}_i Z_i \qquad E(Z_1,...,Z_n)$$

for the same thing: the tensor product of the objects $Z_1,...,Z_n$ of V. The tensor product itself is denoted as E. We denote by E_1 the unary tensor product.

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for the same thing: the tensor product of the objects $Z_1,...,Z_n$ of V. The tensor product itself is denoted as E. We denote by E_1 the unary tensor product.

The additional data for a lax monoidal structure, that is to say a *multitensor*, on V is

$$u_Z: Z \to E_1 Z$$
 $\sigma_{Z_{ij}}: \mathop{\mathsf{E}}_{i} \mathop{\mathsf{E}}_{i} Z_{ij} \to \mathop{\mathsf{E}}_{ii} Z_{ij}$

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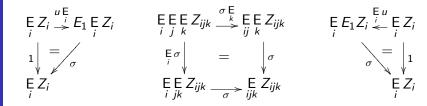
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and this data should be natural in the Z's and satisfy



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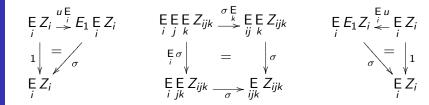
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Relevant Properties

and this data should be natural in the Z's and satisfy



The multitensor is *distributive* when for all n the functor

 $V^n \rightarrow V$ $(X_1, ..., X_n) \mapsto E(X_1, ..., X_n)$

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preserves coproducts in each variable.

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A category enriched in (V, E) consists of $X \in \mathcal{G}V$

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Relevant Properties A category enriched in (V, E) consists of $X \in \mathcal{G}V$ together with composition maps

$$\kappa_{x_i}: \mathop{\mathsf{E}}_{i} X(x_{i-1}, x_i) \to X(x_0, x_n)$$

for all $n \in \mathbb{N}$ and sequences $(x_0, ..., x_n)$ of objects of X, such that

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for all $n \in \mathbb{N}$ and sequences $(x_0, ..., x_n)$ of objects of X, such that

$$\begin{array}{ccc} X(x_0, x_1) \stackrel{u}{\rightarrow} E_1 X(x_0, x_1) & \underset{i \ j}{\mathsf{E}} \mathop{\mathsf{E}}_{i \ j} X(x_{(ij)-1}, x_{ij}) \stackrel{\sigma}{\rightarrow} \mathop{\mathsf{E}}_{ij} X(x_{(ij)-1}, x_{ij}) \\ & & \downarrow^{\kappa} & \underset{i \ k}{\mathsf{E}}_{i \ k} \bigvee & \downarrow^{\kappa} & \\ & & X(x_0, x_1) & \underset{i \ k}{\mathsf{E}} X(x_{(i1)-1}, x_{in_i}) \stackrel{\sigma}{\longrightarrow} X(x_0, x_{mn_m}) \end{array}$$

commute, where $1 \le i \le m$, $1 \le j \le n_i$ and $x_{(11)-1} = x_0$.

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Relevant Properties A category enriched in (V, E) consists of $X \in \mathcal{G}V$ together with composition maps

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commute, where $1 \le i \le m$, $1 \le j \le n_i$ and $x_{(11)-1} = x_0$. We denote by *E*-Cat the category of *E*-categories.

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Example

Let \boldsymbol{V} be a symmetric monoidal category and

 $(A_n: n \in \mathbb{N})$

be the underlying collection of an operad in V.

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Example

Let \boldsymbol{V} be a symmetric monoidal category and

 $(A_n : n \in \mathbb{N})$

be the underlying collection of an operad in V. Then

$$\mathop{\mathsf{E}}_{i} X_{i} := A_{n} \otimes X_{1} \otimes ... \otimes X_{n}$$

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defines a multitensor on V.

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1-object *E*-categories = *A*-algebras

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Example

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defines a multitensor on V.

1-object *E*-categories = *A*-algebras

When V =**Set** this construction gives a bijection between distributive multitensors and non-symmetric operads.

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Example

Let T be a monad on V a category with finite products.

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Example

Let T be a monad on V a category with finite products. Define the multitensor T^{\times} on V by

$$\mathsf{T}_i^{\times} X_i = \prod_i T X_i$$

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Example

Let T be a monad on V a category with finite products. Define the multitensor T^{\times} on V by

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It is distributive when V is and T preserves coproducts

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Example

Let T be a monad on V a category with finite products. Define the multitensor T^{\times} on V by

$$\mathsf{T}_i^{\times} X_i = \prod_i T X_i$$

It is distributive when V is and \mathcal{T} preserves coproducts, and

$$\mathcal{T}^{ imes} ext{-}\mathsf{Cat}\cong(\mathcal{V}^{\mathcal{T}}, imes) ext{-}\mathsf{Cat}$$

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Multitensor to monad construction

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Fundamental Construction

From a distributive multitensor (V, E) one obtains a monad on $\mathcal{G}V$ over **Set**, with underlying endofunctor

$$\Gamma EX(a,b) = \prod_{a=x_0,\dots,x_n=b} \mathsf{E}_i X(x_{i-1},x_i)$$

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and whose category of algebras is E-Cat.

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Examples

One obtains the monads $\mathcal{T}_{\leq n}$ on \mathcal{G}^n **Set** whose algebras are strict *n*-categories as follows:

$$\mathcal{T}_{\leq 0} = 1_{\mathbf{Set}} \qquad \mathcal{T}_{\leq n+1} = \Gamma \mathcal{T}_{\leq n}^{\times}$$

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Examples

One obtains the monads $\mathcal{T}_{\leq n}$ on \mathcal{G}^n **Set** whose algebras are strict *n*-categories as follows:

$$\mathcal{T}_{\leq 0} = 1_{\mathbf{Set}} \qquad \mathcal{T}_{\leq n+1} = \Gamma \mathcal{T}_{< n}^{\times}$$

All the formal categorical properties one knows about $\mathcal{T}_{\leq n}$ can be recovered from general results concerning what \mathcal{G} , Γ and $(-)^{\times}$ preserve.

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Relevant Properties The process

Distributive multitensor on $V \mapsto \mathsf{Monad}$ on $\mathcal{G}V$ over **Set**

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is functorial in some interesting ways.

The process

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Relevant Properties Distributive multitensor on $V \mapsto Monad$ on $\mathcal{G}V$ over **Set**

is functorial in some interesting ways. One can define DISTMULT to be the full sub-2-category of Lax-*M*-Alg whose objects are the distributive lax monoidal categories

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Relevant Properties The process

Distributive multitensor on $V \mapsto Monad$ on $\mathcal{G}V$ over **Set**

is functorial in some interesting ways. One can define DISTMULT to be the full sub-2-category of Lax-M-Alg whose objects are the distributive lax monoidal categories and then one can define a 2-functor

 $\mathsf{\Gamma}:\mathsf{DISTMULT}\to\mathsf{MND}(\mathbf{CAT}/\mathbf{Set})$

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Relevant Properties The process

Distributive multitensor on $V \mapsto Monad$ on $\mathcal{G}V$ over **Set**

is functorial in some interesting ways. One can define DISTMULT to be the full sub-2-category of Lax-M-Alg whose objects are the distributive lax monoidal categories and then one can define a 2-functor

 Γ : DISTMULT \rightarrow MND(**CAT**/**Set**)

Thus in particular, any monoidal monad on a distributive lax monoidal V gives rise to a distributive law between monads defined on $\mathcal{G}V$.

2-functoriality of $\boldsymbol{\Gamma}$

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Relevant Properties Dually, one may define a 2-category OpDISTMULT, whose one-cells are coproduct preserving oplax monoidal functors, and a 2-functor

 Γ : OpDISTMULT \rightarrow OpMND(**CAT**/**Set**)

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Thus any coproduct preserving opmonoidal monad on a distributive lax monoidal V gives rise to a distributive law between monads defined on \mathcal{GV} .

2-functoriality of $\boldsymbol{\Gamma}$

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Example

(E. Cheng) From the inductive description of $\mathcal{T}_{\leq n}$ given above one obtains a distibutive law

$$\mathcal{G}(\mathcal{T}_{\leq n})\Gamma(\prod) \to \Gamma(\prod)\mathcal{G}(\mathcal{T}_{\leq n})$$

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for all *n*, between monads on \mathcal{G}^n **Set**, with composite monad $\Gamma(\prod)\mathcal{G}(\mathcal{T}_{\leq n}) = \mathcal{T}_{\leq (n+1)}$.

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Relevant Properties It is possible to characterise abstractly monads of the form $(\mathcal{GV}, \Gamma E)$. To do this we must introduce two properties of monads: *distributivity* and *path-likeness*.

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Relevant Properties It is possible to characterise abstractly monads of the form $(\mathcal{GV}, \Gamma E)$. To do this we must introduce two properties of monads: *distributivity* and *path-likeness*.

When V has an initial object \emptyset any sequence of objects $(Z_1, ..., Z_n)$ of V may be regarded as a V-graph

$$obj = \{0, ..., n\}$$
 $Hom(i - 1, i) = Z_i$

and all other homs are \emptyset .

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$$obj = \{0, ..., n\}$$
 $Hom(i - 1, i) = Z_i$

and all other homs are \emptyset .

Given a monad T on $\mathcal{G}V$ over **Set** one defines a multitensor

$$\overline{\underline{T}}_i Z_i := T(Z_1, ..., Z_n)(0, n)$$

on V.

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Definition

Let V have coproducts. A monad T on $\mathcal{G}V$ over **Set** is *distributive* when \overline{T} is distributive as a multitensor.

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Relevant Properties

Definition

Let V have coproducts. A monad T on $\mathcal{G}V$ over **Set** is *distributive* when \overline{T} is distributive as a multitensor.

Given an enriched graph X and objects a, b of X, then given coproducts in V one has a canonical map

$$\phi_{X,a,b}: \coprod_{a=x_0,\ldots,x_n=b} \overline{T}_i X(x_{i-1},x_i) \to TX(a,b)$$

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Given an enriched graph X and objects a, b of X, then given coproducts in V one has a canonical map

$$\phi_{X,a,b}: \coprod_{a=x_0,\ldots,x_n=b} \overline{\overline{T}}_i X(x_{i-1},x_i) \to TX(a,b)$$

and one can make

Definition

The monad T is *path-like* when $\phi_{X,a,b}$ is an isomorphism for all X, a and b.

Characterising the image of Γ Lax monoidal categories and higher operads *Note*: if T is path-like then \overline{T} -Cat $\cong \mathcal{G}V^T$. Multitensors and Monads

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Note: if T is path-like then \overline{T} -Cat $\cong \mathcal{G}V^T$.

Theorem

Let V have coproducts. Then a monad T on $\mathcal{G}V$ over **Set** is of the form $(\mathcal{G}V, \Gamma E)$ iff it is

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1 distributive

2 path-like

and in this case E is recaptured as \overline{T} .

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Relevant Properties Given a cartesian monad T, recall that a T-operad is a cartesian monad morphism $\phi : A \to T$. Similarly given a cartesian multitensor E, one may define an E-multitensor to be a cartesian multitensor map into E.

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Corollary

```
Let V be lextensive and E a cartesian multitensor on V. Then \Gamma and \overline{(-)} induce
```

```
E-multitensors \simeq \Gamma E-operads over Set.
```

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Corollary

```
Let V be lextensive and E a cartesian multitensor on V. Then \Gamma and \overline{(-)} induce
```

E-multitensors $\simeq \Gamma E$ -operads over **Set**.

Main case of interest: $E = T_{< n}^{\times}$.

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Relevant Properties Let \mathcal{I} be a class of maps in a category V. Recall that a morphism in V is a *trivial* \mathcal{I} -*fibration* when it satisfies RLP with respect to all elements of \mathcal{I} .

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When V has an initial object any class \mathcal{I} determines a class \mathcal{I}^+ of maps of $\mathcal{G}V$ containing

$$\emptyset \rightarrow ()$$
 $(i): (S) \rightarrow (B)$

where $i \in \mathcal{I}$.

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Relevant Properties Starting with the empty class of maps in **Glob** iterating $(-)^+$ and taking the union, produces the class \mathcal{I}_{∞} containing the inclusion of the boundary of the free-living *n*-cell for each $n \in \mathbb{N}$.

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Relevant Properties Starting with the empty class of maps in **Glob** iterating $(-)^+$ and taking the union, produces the class \mathcal{I}_{∞} containing the inclusion of the boundary of the free-living *n*-cell for each $n \in \mathbb{N}$.

For the finite dimensional versions – classes in \mathcal{G}^n **Set** denoted $\mathcal{I}_{\leq n}$ – start with the class

$$\{ \emptyset \to 1, \, 2 \to 1 \}$$

in **Set** and successively apply $(-)^+$.

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$$\{ \emptyset \to 1, \, 2 \to 1 \}$$

in **Set** and successively apply $(-)^+$.

Leinster: An n-operad is *contractible* when it is a trivial $\mathcal{I}_{\leq n}$ -fibration.

Contractible multitensors and operads



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Relevant Properties

Proposition

Let *E* be a $\mathcal{T}_{\leq n}$ -multitensor. Then *E* is a contractible *n*-multitensor iff ΓE is a contractible (n+1)-operad.

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Relevant Properties There are two basic ingredients for the Trimble definition. The first is the path space functor

 $\mathbb{P}: \textbf{Top} \to \mathcal{G}\textbf{Top}$

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and this arises quite canonically in our setting.

Background from topology

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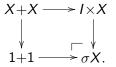
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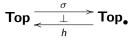
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Relevant Properties For any space X the reduced suspension of X, is defined as the pushout



and so we get



where h(a, X, b) is the space of paths in X from a to b.

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Free Products

Relevant Properties On the categorical side one has the 2-adjunction

$$CAT/Set \xrightarrow{(-)_{\bullet}}_{\leq \perp} CAT$$

where for $f : A \rightarrow \mathbf{Set}$, A_{\bullet} is the category of bipointed objects in A.

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where for $f : A \rightarrow \mathbf{Set}$, A_{\bullet} is the category of bipointed objects in A.

The path space functor \mathbb{P} is the composite

$$\mathsf{Top} \xrightarrow{\mathsf{unit}} \mathcal{G} \mathsf{Top}_{\bullet} \xrightarrow{\mathcal{G}h} \mathcal{G} \mathsf{Top}$$

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$$CAT/Set \xrightarrow{(-)_{\bullet}}_{\leq \perp} CAT$$

where for $f : A \rightarrow \mathbf{Set}$, A_{\bullet} is the category of bipointed objects in A.

The path space functor \mathbb{P} is the composite



and from this description $\ensuremath{\mathbb{P}}$ is evidently a right adjoint.

Operad action on path-spaces

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Free Products

Relevant Properties The second basic ingredient for the Trimble definition is a non-symmetric contractible topological operad *A* which acts on the path space functor.

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Relevant Properties The second basic ingredient for the Trimble definition is a non-symmetric contractible topological operad A which acts on the path space functor. To say that A acts on \mathbb{P} is to say that \mathbb{P} factors as

$$\mathbf{Top} \xrightarrow{P_A} A\operatorname{-Cat} \xrightarrow{U^A} \mathcal{G} \mathbf{Top}$$

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Relevant Properties The second basic ingredient for the Trimble definition is a non-symmetric contractible topological operad A which acts on the path space functor. To say that A acts on \mathbb{P} is to say that \mathbb{P} factors as

$$\mathbf{Fop} \xrightarrow{P_A} A - \mathsf{Cat} \xrightarrow{U^A} \mathcal{G} \mathbf{Top}$$

Since \mathbb{P} is a right adjoint, P_A is also a right adjoint by the Dubuc adjoint triangle theorem.

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Relevant Properties

$$(Q,V)\mapsto (Q^{(+)},V^{(+)})$$

Applies to pairs (Q, V) consisting of a distributive category V and a product preserving $Q : \mathbf{Top} \to V$.

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Relevant Properties

$$(Q,V)\mapsto (Q^{(+)},V^{(+)})$$

Applies to pairs (Q, V) consisting of a distributive category V and a product preserving $Q : \mathbf{Top} \to V$.

Regarding Q as a (strong) monoidal functor (**Top**, A) \rightarrow (V, QA), and applying Γ gives a monad morphism

 $(\mathcal{G}(\textbf{Top}), \Gamma(A)) \to (\mathcal{G}V, \Gamma(QA))$

with underlying functor $\mathcal{G}Q$.

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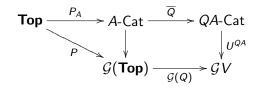
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Relevant Properties By formal monad theory this monad morphism amounts to giving a lifting \overline{Q} as indicated in the commutative diagram



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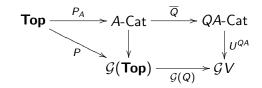
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Relevant Properties By formal monad theory this monad morphism amounts to giving a lifting \overline{Q} as indicated in the commutative diagram



and then one defines

$$Q^{(+)} = \overline{Q}P_A$$
 $V^{(+)} = QA$ -Cat

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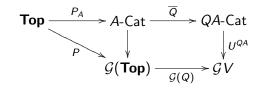
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Free Products

Relevant Properties By formal monad theory this monad morphism amounts to giving a lifting \overline{Q} as indicated in the commutative diagram



and then one defines

 $Q^{(+)} = \overline{Q}P_A$ $V^{(+)} = QA$ -Cat

Iterating this starting from π_0 : **Top** \rightarrow **Set** produces the fundamental *n*-groupoid functor from **Top** into the category of Trimble *n*-categories.

Cheng's theorem

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Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

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(indication of proof):

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Relevant Properties

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

(*indication of proof*): The monads Trm_n are given inductively as

$$\operatorname{Trm}_0 = 1_{\operatorname{Set}} \qquad \operatorname{Trm}_{n+1} = \Gamma(\pi_n A)$$

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by construction,

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$$\mathsf{Trm}_0 = 1_{\mathsf{Set}}$$
 $\mathsf{Trm}_{n+1} = \mathsf{F}(\pi_n A)$

by construction, and the cartesian multitensor map $A \rightarrow \prod$ can be used to construct the *n*-operad structure.

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by construction, and the cartesian multitensor map $A \to \prod$ can be used to construct the *n*-operad structure. Contractibility in the topological setting and contractibility in the globular setting are compatible because of how \mathbb{P} arises,

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 $\mathsf{Trm}_{n+1} = \Gamma(\pi_n A)$

by construction, and the cartesian multitensor map $A \to \prod$ can be used to construct the *n*-operad structure. Contractibility in the topological setting and contractibility in the globular setting are compatible because of how \mathbb{P} arises, and this together with the general theory enables one to verify the contractibility of Trm_n.

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Relevant Properties Recall that given an (n+1)-operad A over **Set**, \overline{A} is an *n*-multitensor, so in particular gives a lax monoidal structure on \mathcal{G}^n **Set**, and \overline{A} -categories are the same as A-algebras.

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Example

Let A be the 3-operad for Gray categories. Then \overline{A} is a lax monoidal structure on 2-globular sets and \overline{A} -categories are Gray categories. Note that \overline{A}_1 is the monad for (strict) 2-categories.

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Example

Let A be the 3-operad for Gray categories. Then A is a lax monoidal structure on 2-globular sets and \overline{A} -categories are Gray categories. Note that \overline{A}_1 is the monad for (strict) 2-categories.

Question: how are \overline{A} and the Gray tensor product for 2-categories related?

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Relevant Properties

Let (V, E) be a lax monoidal category. Then (E_1, u, σ) is a

monad and

Lemma

1 for any $(X_1, ..., X_n)$, $\underset{i}{\mathsf{E}} X_i$ is an E_1 -algebra.

2 for any E-category X, its homs are E_1 -algebras.

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Lemma

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Relevant Properties Let (V, E) be a lax monoidal category. Then (E_1, u, σ) is a monad and

1 for any $(X_1, ..., X_n)$, $\underset{i}{\mathsf{E}} X_i$ is an E_1 -algebra.

2 for any *E*-category X, its homs are E_1 -algebras.

Question: is there a canonical way to lift *E* to a multitensor *E'* on V^{E_1} so that *E'*-Cat \cong *E*-Cat?

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Free Products

Relevant Properties

Theorem

Let E be an accessible distributive multitensor on a locally presentable category V. Then there is, to within isomorphism, a unique multitensor E' on V^{E_1} such that

1 the unit for E' is the identity.

2 E' is distributive.

 $\mathbf{3}$ E'-Cat \cong E-Cat.

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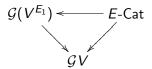
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Free Products

Relevant Properties *Indication of the proof*: One has a commutative triangle of forgetful functors



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and so an induced monad T on $\mathcal{G}(V^{E_1})$.

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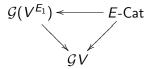
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Free Products

Relevant Properties *Indication of the proof*: One has a commutative triangle of forgetful functors



and so an induced monad T on $\mathcal{G}(V^{E_1})$. This monad turns out to be distributive and path-like and so one can take $E' = \overline{T}$.

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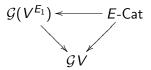
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For uniqueness let F be some multitensor with the desired properties.

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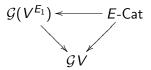
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For uniqueness let F be some multitensor with the desired properties. Then ΓF and T are monads on $\mathcal{G}(V^{E_1})$ with the same algebras,

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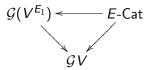
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and so an induced monad T on $\mathcal{G}(V^{E_1})$. This monad turns out to be distributive and path-like and so one can take $E' = \overline{T}$.

For uniqueness let F be some multitensor with the desired properties. Then ΓF and T are monads on $\mathcal{G}(V^{E_1})$ with the same algebras, thus they're isomorphic and so $F \cong \overline{T}$.

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Free Products

Relevant Properties Recall that if you take the Gray tensor product of 2-categories, and observe what happens in below dimension 2, that one is observing a canonical tensor product of categories.

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Free Products

Relevant Properties Recall that if you take the Gray tensor product of 2-categories, and observe what happens in below dimension 2, that one is observing a canonical tensor product of categories.

Theorem

(Flotz, Kelly and Lair) Up to isomorphism there are exactly two biclosed monoidal structures on **Cat**, both symmetric.

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Theorem

(Flotz, Kelly and Lair) Up to isomorphism there are exactly two biclosed monoidal structures on **Cat**, both symmetric.

The "other one" is often called the "funny" tensor product, but we call it and its generalisations *free products*.

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Free Products

Relevant Properties • there's an identity on objects comparison $A \otimes B \rightarrow A \times B$ natural in A and B.

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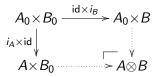
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Free Products

Relevant Properties

- there's an identity on objects comparison A ⊗ B → A × B natural in A and B.
- one has the *pushout formula*:



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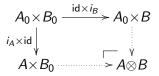
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Free Products

Relevant Properties

- there's an identity on objects comparison $A \otimes B \rightarrow A \times B$ natural in A and B.
- one has the *pushout formula*:



• explicitly, morphisms of $A \otimes B$ are generated by

 $(a,\beta):(a,b_1) \rightarrow (a,b_2)$ $(\alpha,b):(a_1,b) \rightarrow (a_2,b)$

subject to relations remembering composition in A and B.

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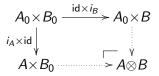
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Relevant Properties

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 $(a,\beta):(a,b_1) \rightarrow (a,b_2)$ $(\alpha,b):(a_1,b) \rightarrow (a_2,b)$

subject to relations remembering composition in A and B. [A, B] consists of functors and "transformations".

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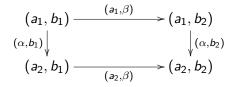
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Free Products

Relevant Properties For maps α and β the square



commutes in the cartesian product, but **not** in the free product.

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For maps lpha and eta the square

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Free Products

 $\begin{array}{c} (a_1, b_1) \xrightarrow{(a_1, \beta)} (a_1, b_2) \\ (\alpha, b_1) \downarrow & \downarrow (\alpha, b_2) \\ (a_2, b_1) \xrightarrow{(a_2, \beta)} (a_2, b_2) \end{array}$

commutes in the cartesian product, but **not** in the free product.

The Gray tensor product proceeds in the same way for objects and arrows, and in dimension 2 one has a coherent isomorphism between the two diagonals of this square.

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Free Products

Relevant Properties For any *n*-operad over **Set** there is an analogue of the free product for its algebras.

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Free Products

Relevant Properties

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathsf{SMItCAT} \xleftarrow{\mathcal{U}} \mathsf{SMonCAT}$$

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Free Products

Relevant Properties $\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathsf{SMItCAT} \xleftarrow{\mathcal{U}} \mathsf{SMonCAT}$

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 $(\mathcal{G}V,T)$

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Free Products

Relevant Properties

$$CAT/Set \xrightarrow{\mathcal{F}} SMItCAT \xleftarrow{\mathcal{U}} SMonCAT$$
$$(\mathcal{G}V, \mathcal{T}) \longmapsto (\mathcal{F}\mathcal{G}V, \mathcal{F}\mathcal{T})$$

To be described: ${\mathcal F}$ is an EM-object-preserving 2-functor over ${\rm CAT}.$

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$$CAT/Set \xrightarrow{\mathcal{F}} SMItCAT \xleftarrow{\mathcal{U}} SMonCAT$$
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Hermida: $\ensuremath{\mathcal{U}}$ is 2-fully-faithful and its image consists of the representable multicategories.

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$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathsf{SMItCAT} \xleftarrow{\mathcal{U}} \mathsf{SMonCAT}$$
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Hermida: \mathcal{U} is 2-fully-faithful and its image consists of the representable multicategories. Under mild conditions on V the multicategory \mathcal{FGV} is representable and closed (in the sense of Manzyuk).

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To be described: \mathcal{F} is an EM-object-preserving 2-functor over **CAT**.

Hermida: \mathcal{U} is 2-fully-faithful and its image consists of the representable multicategories. Under mild conditions on V the multicategory \mathcal{FGV} is representable and closed (in the sense of Manzyuk). By the 2-fully-faithfulness of \mathcal{U} , T is automatically a symmetric monoidal monad.

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Relevant Properties

(A. Kock) Let T be a symmetric monoidal monad on V a symmetric monoidal closed category with equalisers. Then the equaliser

$$[(X, x), (Y, y)] \longrightarrow [X, Y] \xrightarrow{[x, \text{id}]} [TX, Y]$$
$$[TX, TY]$$

defines the internal hom and (TI, μ_I) the unit of a closed structure on V^T .

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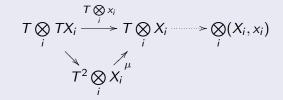
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Relevant Properties (F. Linton, B. Day) If in addition V^T has coequalisers then the coequaliser



defines the associated tensor product on V^T making it symmetric monoidal closed.

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Relevant Properties

Definition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. The tensor product on $\mathcal{G}V^T$ is called the *free* product of T-algebras.

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Relevant Properties Let A be a category over **Set** and for $a \in A$, denote by a_0 the underlying set of a.

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Relevant Properties Let *A* be a category over **Set** and for $a \in A$, denote by a_0 the underlying set of *a*. A multimap of $\mathcal{F}A$

$$f:(a_1,...,a_n)\to b$$

consists of a function

 $f_0: a_{1,0} \times ... \times a_{n,0} \rightarrow b_0$

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together with for each $1 \le i^* \le n$ and $z \in \prod_{i \ne i^*} a_{i,0}$, a morphism $f_z : a_{i^*} \to b$ of A such that $(f_z)_0 = (f_0)_z$.

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 $f_z: a_{i^*} \to b$ of A such that $(f_z)_0 = (f_0)_z$.

A nullary multimap is just an object of b and a linear map is just a morphism of A.

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Example

 $\mathcal{F}R$ -Mod is the symmetric multicategory of R-modules and R-multilinear maps.

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 $\mathcal{F}R$ -Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Example

Let X be a set and M the monoid of endofunctions of X.

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 $\mathcal{F}R$ -Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Example

Let X be a set and M the monoid of endofunctions of X. The monoid M acts on X by evaluation giving a functor $M \rightarrow \mathbf{Set}$.

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Let X be a set and M the monoid of endofunctions of X. The monoid M acts on X by evaluation giving a functor $M \rightarrow \mathbf{Set}$. The symmetric multicategory $\mathcal{F}M$ has one object, thus it is just an operad of sets. In fact $\mathcal{F}M$ is the endomorphism operad of X.

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Relevant Properties Bourke: the Gray tensor product for 2-categories can be obtained by factoring the canonical map from the free to the cartesian product.

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Free Products

Relevant Properties Bourke: the Gray tensor product for 2-categories can be obtained by factoring the canonical map from the free to the cartesian product.

There is no such map for mere graphs. So one can ask: when is there such a canonical map?

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Free Products

Relevant Properties The unit for the free product on $\mathcal{G}V$ is denoted 0: the V-graph with one object and initial hom. Given a monad T on $\mathcal{G}V$, the unit for the free product on $\mathcal{G}V^T$ is $(T0, \mu_0)$.

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If we had

 $A \otimes B \to A \times B$

then putting A = 1 and $B = (T0, \mu_0)$ would give a map $e: 1 \rightarrow (T0, \mu_0).$

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It is obviously a split monomorphism

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It is obviously a split monomorphism. In fact it's an isomorphism

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 $A \otimes B \to A \times B$

then putting A = 1 and $B = (T0, \mu_0)$ would give a map $e: 1 \rightarrow (T0, \mu_0).$

It is obviously a split monomorphism. In fact it's an isomorphism: to see that the composite

$$T0 \longrightarrow 1 \xrightarrow{e} T0$$

is the identity it suffices that the composite morphism

$$0 \xrightarrow{\eta_0} T 0 \longrightarrow 1 \xrightarrow{e} T 0$$

is η_0 .

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Relevant Properties

Proposition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. There is an identity on objects morphism $\otimes \to \times$ of tensor products iff T0 = 1.

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Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. There is an identity on objects morphism $\otimes \to \times$ of tensor products iff T0 = 1.

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Such monads T are said to be *well-pointed*.

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Proposition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. There is an identity on objects morphism $\otimes \to \times$ of tensor products iff T0 = 1.

Such monads T are said to be *well-pointed*.

When T is coproduct preserving and satisfies the conditions of the proposition, the free product of T-algebras satisfies the pushout formula.

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Relevant Properties

A sesqui-T-algebra is a category enriched in $\mathcal{G}V^T$ for the free product.

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Relevant Properties A *sesqui-T-algebra* is a category enriched in \mathcal{GV}^T for the free product.

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Question: if T is an *n*-operad, then are sesqui-T-algebras (n+1)-operadic?

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Relevant Properties A sesqui-T-algebra is a category enriched in \mathcal{GV}^T for the free product.

Question: if T is an *n*-operad, then are sesqui-T-algebras (n+1)-operadic?

Theorem

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible and coproduct preserving. Then the monad on \mathcal{G}^2V whose algebras are sesqui-T-algebras is given explicitly as $\Gamma(T \otimes)$.

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Theorem

Let V be locally presentable and extensive, and T be a monad on $\mathcal{G}V$ over **Set** which is accessible, well-pointed, l.r.a, distributive and path-like. Suppose that $\psi : A \rightarrow T$ is a T-operad. Then

$$A\bigotimes_{i} X_{i} \xrightarrow{\psi_{\bigotimes_{i}} X_{i}} T\bigotimes_{i} X_{i} \longrightarrow \prod_{i} TX_{i}$$

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are the components of a T^{\times} -multitensor.