Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Lax monoidal categories and higher operads

Mark Weber

CT2010 Genova June 2010

joint in part with
Michael Batanin and Denis-Charles Cisinski

For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

$$\text{objects} : \mathcal{G}V \to \mathbf{Set}$$

For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

$$\text{objects} : \mathcal{G}V \to \mathbf{Set}$$

In particular

$$\mathcal{G}1 = \mathbf{Set} \qquad \mathcal{G}\mathbf{Set} = \mathbf{Graph}$$

the category of n-globular sets is $\mathcal{G}^n\mathbf{Set}$, and $\mathcal{G}\mathbf{Glob} \cong \mathbf{Glob}$.

For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

$$\text{objects} : \mathcal{G}V \to \mathbf{Set}$$

In particular

$$\mathcal{G}1 = \mathbf{Set} \qquad \mathcal{G}\mathbf{Set} = \mathbf{Graph}$$

the category of n-globular sets is $\mathcal{G}^n\mathbf{Set}$, and $\mathcal{G}\mathbf{Glob} \cong \mathbf{Glob}$.

Batanin: higher categorical structures are algebras of monads on $\mathcal{G}^n\mathbf{Set}$ or $\mathbf{Glob}$.

For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

$$\text{objects} : \mathcal{G}V \to \mathbf{Set}$$

In particular

$$\mathcal{G}1 = \mathbf{Set} \qquad \mathcal{G}\mathbf{Set} = \mathbf{Graph}$$

the category of n-globular sets is $\mathcal{G}^n\mathbf{Set}$, and $\mathcal{G}\mathbf{Glob} \cong \mathbf{Glob}$.

Batanin: higher categorical structures are algebras of monads on $\mathcal{G}^n\mathbf{Set}$ or $\mathbf{Glob}$.

Useful abstraction: globular higher category theory is about monads defined on $\mathcal{G}V$ over $\mathbf{Set}$.

For any category V one may consider the category $\mathcal{G}V$ of graphs enriched in V.

$$\text{objects} : \mathcal{G}V \to \mathbf{Set}$$

In particular

$$\mathcal{G}1 = \mathbf{Set} \qquad \mathcal{G}\mathbf{Set} = \mathbf{Graph}$$

the category of n-globular sets is $\mathcal{G}^n\mathbf{Set}$, and $\mathcal{G}\mathbf{Glob} \cong \mathbf{Glob}$.

Batanin: higher categorical structures are algebras of monads on $\mathcal{G}^n\mathbf{Set}$ or $\mathbf{Glob}$.

Useful abstraction: globular higher category theory is about monads defined on $\mathcal{G}V$ over $\mathbf{Set}$.The monads of interest come from distributive lax monoidal structures on V.

Outline

1 Distributive multitensors and monads

Outline

1 Distributive multitensors and monads

2 Contractible operads and contractible multitensors

Outline

1 Distributive multitensors and monads

2 Contractible operads and contractible multitensors

3 The Trimble definition á la Cheng

Outline

1 Distributive multitensors and monads

2 Contractible operads and contractible multitensors

3 The Trimble definition á la Cheng

4 Gray and Crans tensor products from operads

Outline

1 Distributive multitensors and monads

2 Contractible operads and contractible multitensors

3 The Trimble definition á la Cheng

4 Gray and Crans tensor products from operads

5 Free products of higher categories

Outline

1. Distributive multitensors and monads
2. Contractible operads and contractible multitensors
3. The Trimble definition á la Cheng
4. Gray and Crans tensor products from operads
5. Free products of higher categories
6. Properties of free products relevant for an inductive definition of semi-strict n-category

Multitensors

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Let V be a category. We shall use the following notations

$$\underset{1 \leq i \leq n}{\mathrm{E}} Z_i \qquad \underset{i}{\mathrm{E}} Z_i \qquad E(Z_1, ..., Z_n)$$

for the same thing: the tensor product of the objects $Z_1,...,Z_n$ of V. The tensor product itself is denoted as E. We denote by E_1 the unary tensor product.

Multitensors

Let V be a category. We shall use the following notations

$$\underset{1 \leq i \leq n}{E} Z_i \qquad \underset{i}{E} Z_i \qquad E(Z_1, ..., Z_n)$$

for the same thing: the tensor product of the objects $Z_1,...,Z_n$ of V. The tensor product itself is denoted as E. We denote by E_1 the unary tensor product.

The additional data for a lax monoidal structure, that is to say a *multitensor*, on V is

$$u_Z : Z \to E_1 Z \qquad \sigma_{Z_{ij}} : \underset{i}{E} \underset{j}{E} Z_{ij} \to \underset{ij}{E} Z_{ij}$$

Multitensors

and this data should be natural in the Z's and satisfy

$$\begin{array}{ccc} \underset{i}{\mathsf{E}}\, Z_i & \xrightarrow{u\,\underset{i}{\mathsf{E}}} & E_1 \underset{i}{\mathsf{E}}\, Z_i \\ {\scriptstyle 1}\downarrow & = & \swarrow{\scriptstyle \sigma} \\ \underset{i}{\mathsf{E}}\, Z_i & & \end{array} \qquad \begin{array}{ccc} \underset{i}{\mathsf{E}}\, \underset{j}{\mathsf{E}}\, \underset{k}{\mathsf{E}}\, Z_{ijk} & \xrightarrow{\sigma\,\underset{k}{\mathsf{E}}} & \underset{ij}{\mathsf{E}}\, \underset{k}{\mathsf{E}}\, Z_{ijk} \\ {\scriptstyle \underset{i}{\mathsf{E}}\,\sigma}\downarrow & = & \downarrow{\scriptstyle \sigma} \\ \underset{i}{\mathsf{E}}\, \underset{jk}{\mathsf{E}}\, Z_{ijk} & \xrightarrow[\sigma]{} & \underset{ijk}{\mathsf{E}}\, Z_{ijk} \end{array} \qquad \begin{array}{ccc} \underset{i}{\mathsf{E}}\, E_1 Z_i & \xleftarrow{\underset{i}{\mathsf{E}}\,u} & \underset{i}{\mathsf{E}}\, Z_i \\ & {\scriptstyle \sigma}\searrow \quad = & \downarrow{\scriptstyle 1} \\ & & \underset{i}{\mathsf{E}}\, Z_i \end{array}$$

Multitensors

and this data should be natural in the Z's and satisfy

$$\begin{array}{ccc} \underset{i}{\mathrm{E}}\, Z_i & \xrightarrow{u\,\underset{i}{\mathrm{E}}} & E_1 \underset{i}{\mathrm{E}}\, Z_i \\ {\scriptstyle 1}\downarrow & = & \swarrow{\scriptstyle \sigma} \\ \underset{i}{\mathrm{E}}\, Z_i & & \end{array} \qquad \begin{array}{ccc} \underset{i}{\mathrm{E}}\,\underset{j}{\mathrm{E}}\,\underset{k}{\mathrm{E}}\, Z_{ijk} & \xrightarrow{\sigma\,\underset{k}{\mathrm{E}}} & \underset{ij}{\mathrm{E}}\,\underset{k}{\mathrm{E}}\, Z_{ijk} \\ {\scriptstyle \underset{i}{\mathrm{E}}\,\sigma}\downarrow & = & \downarrow{\scriptstyle \sigma} \\ \underset{i}{\mathrm{E}}\,\underset{jk}{\mathrm{E}}\, Z_{ijk} & \xrightarrow[\sigma]{} & \underset{ijk}{\mathrm{E}}\, Z_{ijk} \end{array} \qquad \begin{array}{ccc} \underset{i}{\mathrm{E}}\, E_1 Z_i & \xleftarrow{\underset{i}{\mathrm{E}}\,u} & \underset{i}{\mathrm{E}}\, Z_i \\ & {\scriptstyle \sigma}\searrow \;\; = & \downarrow{\scriptstyle 1} \\ & & \underset{i}{\mathrm{E}}\, Z_i \end{array}$$

The multitensor is *distributive* when for all n the functor

$$V^n \to V \qquad (X_1, ..., X_n) \mapsto E(X_1, ..., X_n)$$

preserves coproducts in each variable.

Categories enriched in a multitensor

A category enriched in (V, E) consists of $X \in \mathcal{G}V$

Categories enriched in a multitensor

A category enriched in (V, E) consists of $X \in \mathcal{G}V$ together with composition maps

$$\kappa_{x_i} : \underset{i}{E}\, X(x_{i-1}, x_i) \to X(x_0, x_n)$$

for all $n \in \mathbb{N}$ and sequences $(x_0, ..., x_n)$ of objects of X, such that

Categories enriched in a multitensor

A category enriched in (V, E) consists of $X \in \mathcal{G}V$ together with composition maps

$$\kappa_{x_i} : \underset{i}{\mathrm{E}}\, X(x_{i-1}, x_i) \to X(x_0, x_n)$$

for all $n \in \mathbb{N}$ and sequences $(x_0, ..., x_n)$ of objects of X, such that

$$\begin{array}{ccc} X(x_0, x_1) & \xrightarrow{u} & E_1 X(x_0, x_1) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \kappa} \\ & & X(x_0, x_1) \end{array} \qquad \begin{array}{ccc} \underset{i}{\mathrm{E}}\, \underset{j}{\mathrm{E}}\, X(x_{(ij)-1}, x_{ij}) & \xrightarrow{\sigma} & \underset{ij}{\mathrm{E}}\, X(x_{(ij)-1}, x_{ij}) \\ {\scriptstyle \underset{i}{\mathrm{E}}\,\kappa}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \underset{i}{\mathrm{E}}\, X(x_{(i1)-1}, x_{in_i}) & \xrightarrow[\kappa]{} & X(x_0, x_{mn_m}) \end{array}$$

commute, where $1 \leq i \leq m$, $1 \leq j \leq n_i$ and $x_{(11)-1} = x_0$.

Categories enriched in a multitensor

A category enriched in (V, E) consists of $X \in \mathcal{G}V$ together with composition maps

$$\kappa_{x_i} : \underset{i}{E}\, X(x_{i-1}, x_i) \to X(x_0, x_n)$$

for all $n \in \mathbb{N}$ and sequences $(x_0, ..., x_n)$ of objects of X, such that

$$\begin{array}{ccc} X(x_0, x_1) & \xrightarrow{u} & E_1 X(x_0, x_1) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \kappa} \\ & & X(x_0, x_1) \end{array} \qquad \begin{array}{ccc} \underset{i}{E}\,\underset{j}{E}\, X(x_{(ij)-1}, x_{ij}) & \xrightarrow{\sigma} & \underset{ij}{E}\, X(x_{(ij)-1}, x_{ij}) \\ {\scriptstyle \underset{i}{E}\kappa}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \underset{i}{E}\, X(x_{(i1)-1}, x_{in_i}) & \xrightarrow[\kappa]{} & X(x_0, x_{mn_m}) \end{array}$$

commute, where $1 \le i \le m$, $1 \le j \le n_i$ and $x_{(11)-1} = x_0$.
We denote by E-Cat the category of E-categories.

Example

Let V be a symmetric monoidal category and

$$(A_n : n \in \mathbb{N})$$

be the underlying collection of an operad in V.

Example

Let V be a symmetric monoidal category and

$$(A_n : n \in \mathbb{N})$$

be the underlying collection of an operad in V. Then

$$\underset{i}{E}\, X_i := A_n \otimes X_1 \otimes \ldots \otimes X_n$$

defines a multitensor on V.

Example

Let V be a symmetric monoidal category and

$$(A_n : n \in \mathbb{N})$$

be the underlying collection of an operad in V. Then

$$\underset{i}{\mathrm{E}}\, X_i := A_n \otimes X_1 \otimes \ldots \otimes X_n$$

defines a multitensor on V.

$$\text{1-object } E\text{-categories} \quad = \quad A\text{-algebras}$$

Example

Let V be a symmetric monoidal category and

$$(A_n : n \in \mathbb{N})$$

be the underlying collection of an operad in V. Then

$$\underset{i}{\mathrm{E}}\, X_i := A_n \otimes X_1 \otimes \ldots \otimes X_n$$

defines a multitensor on V.

$$\text{1-object } E\text{-categories} \quad = \quad A\text{-algebras}$$

When $V = \mathbf{Set}$ this construction gives a bijection between distributive multitensors and non-symmetric operads.

Example

Let T be a monad on V a category with finite products.

Example

Let T be a monad on V a category with finite products. Define the multitensor $T^{\times}$ on V by

$$\underset{i}{\mathrm{T}}^{\times} X_i = \prod_i TX_i$$

Example

Let T be a monad on V a category with finite products. Define the multitensor $T^{\times}$ on V by

$$\mathrm{T}^{\times}_{i} X_i = \prod_i TX_i$$

It is distributive when V is and T preserves coproducts

Example

Let T be a monad on V a category with finite products. Define the multitensor $T^{\times}$ on V by

$$\underset{i}{\mathrm{T}^{\times}} X_i = \prod_i TX_i$$

It is distributive when V is and T preserves coproducts, and

$$T^{\times}\text{-Cat} \cong (V^T, \times)\text{-Cat}$$

Multitensor to monad construction

Fundamental Construction

From a distributive multitensor (V, E) *one obtains a monad on* $\mathcal{G}V$ *over* **Set**, *with underlying endofunctor*

$$\Gamma EX(a,b) = \coprod_{a=x_0,\ldots,x_n=b} \underset{i}{\mathrm{E}}\, X(x_{i-1}, x_i)$$

and whose category of algebras is E-Cat.

Strict n-category monads made easy

Examples

One obtains the monads $\mathcal{T}_{\leq n}$ on $\mathcal{G}^n\mathbf{Set}$ whose algebras are strict n-categories as follows:

$$\mathcal{T}_{\leq 0} = 1_{\mathbf{Set}} \qquad \mathcal{T}_{\leq n+1} = \Gamma \mathcal{T}^{\times}_{\leq n}$$

Strict n-category monads made easy

Examples

One obtains the monads $\mathcal{T}_{\leq n}$ on $\mathcal{G}^n\mathbf{Set}$ whose algebras are strict n-categories as follows:

$$\mathcal{T}_{\leq 0} = 1_{\mathbf{Set}} \qquad \mathcal{T}_{\leq n+1} = \Gamma\mathcal{T}_{\leq n}^{\times}$$

All the formal categorical properties one knows about $\mathcal{T}_{\leq n}$ can be recovered from general results concerning what $\mathcal{G}$, Γ and $(-)^{\times}$ preserve.

2-functoriality of Γ

The process

$$\text{Distributive multitensor on } V \mapsto \text{Monad on } \mathcal{G}V \text{ over } \mathbf{Set}$$

is functorial in some interesting ways.

2-functoriality of Γ

The process

$$\text{Distributive multitensor on } V \mapsto \text{Monad on } \mathcal{G}V \text{ over } \mathbf{Set}$$

is functorial in some interesting ways. One can define DISTMULT to be the full sub-2-category of Lax-M-Alg whose objects are the distributive lax monoidal categories

2-functoriality of Γ

The process

$$\text{Distributive multitensor on } V \mapsto \text{Monad on } \mathcal{G}V \text{ over } \mathbf{Set}$$

is functorial in some interesting ways. One can define DISTMULT to be the full sub-2-category of Lax-M-Alg whose objects are the distributive lax monoidal categories and then one can define a 2-functor

$$\Gamma : \text{DISTMULT} \to \text{MND}(\mathbf{CAT}/\mathbf{Set})$$

2-functoriality of Γ

The process

$$\text{Distributive multitensor on } V \mapsto \text{Monad on } \mathcal{G}V \text{ over } \mathbf{Set}$$

is functorial in some interesting ways. One can define DISTMULT to be the full sub-2-category of Lax-M-Alg whose objects are the distributive lax monoidal categories and then one can define a 2-functor

$$\Gamma : \text{DISTMULT} \to \text{MND}(\mathbf{CAT}/\mathbf{Set})$$

Thus in particular, any monoidal monad on a distributive lax monoidal V gives rise to a distributive law between monads defined on $\mathcal{G}V$.

2-functoriality of Γ

Dually, one may define a 2-category OpDISTMULT, whose one-cells are coproduct preserving oplax monoidal functors, and a 2-functor

$$\Gamma : \text{OpDISTMULT} \to \text{OpMND}(\mathbf{CAT}/\mathbf{Set})$$

Thus any coproduct preserving opmonoidal monad on a distributive lax monoidal V gives rise to a distributive law between monads defined on $\mathcal{G}V$.

2-functoriality of Γ

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Example

(E. Cheng) From the inductive description of $\mathcal{T}_{\leq n}$ given above one obtains a distibutive law

$$\mathcal{G}(\mathcal{T}_{\leq n})\Gamma(\prod) \to \Gamma(\prod)\mathcal{G}(\mathcal{T}_{\leq n})$$

for all n, between monads on $\mathcal{G}^n\mathbf{Set}$, with composite monad $\Gamma(\prod)\mathcal{G}(\mathcal{T}_{\leq n}) = \mathcal{T}_{\leq (n+1)}$.

Characterising the image of Γ

It is possible to characterise abstractly monads of the form $(\mathcal{G}V, \Gamma E)$. To do this we must introduce two properties of monads: *distributivity* and *path-likeness*.

Characterising the image of Γ

It is possible to characterise abstractly monads of the form $(\mathcal{G}V, \Gamma E)$. To do this we must introduce two properties of monads: *distributivity* and *path-likeness*.

When V has an initial object $\emptyset$ any sequence of objects $(Z_1, ..., Z_n)$ of V may be regarded as a V-graph

$$\text{obj} = \{0, ..., n\} \qquad \text{Hom}(i-1, i) = Z_i$$

and all other homs are $\emptyset$.

Characterising the image of Γ

It is possible to characterise abstractly monads of the form $(\mathcal{G}V, \Gamma E)$. To do this we must introduce two properties of monads: *distributivity* and *path-likeness*.

When V has an initial object $\emptyset$ any sequence of objects $(Z_1, ..., Z_n)$ of V may be regarded as a V-graph

$$\text{obj} = \{0, ..., n\} \qquad \text{Hom}(i-1, i) = Z_i$$

and all other homs are $\emptyset$.

Given a monad T on $\mathcal{G}V$ over **Set** one defines a multitensor

$$\overline{T}_i Z_i := T(Z_1, ..., Z_n)(0, n)$$

on V.

Characterising the image of Γ

Definition

Let V have coproducts. A monad T on $\mathcal{G}V$ over **Set** is *distributive* when $\overline{T}$ is distributive as a multitensor.

Characterising the image of Γ

Definition

Let V have coproducts. A monad T on $\mathcal{G}V$ over **Set** is *distributive* when $\overline{T}$ is distributive as a multitensor.

Given an enriched graph X and objects a, b of X, then given coproducts in V one has a canonical map

$$\phi_{X,a,b} : \coprod_{a=x_0,\ldots,x_n=b} \overline{T}_i X(x_{i-1}, x_i) \to TX(a,b)$$

Characterising the image of Γ

Definition

Let V have coproducts. A monad T on $\mathcal{G}V$ over **Set** is *distributive* when $\overline{T}$ is distributive as a multitensor.

Given an enriched graph X and objects a, b of X, then given coproducts in V one has a canonical map

$$\phi_{X,a,b} : \coprod_{a=x_0,\ldots,x_n=b} \overline{T}_i X(x_{i-1}, x_i) \to TX(a,b)$$

and one can make

Definition

The monad T is *path-like* when $\phi_{X,a,b}$ is an isomorphism for all X, a and b.

Characterising the image of Γ

Note: if T is path-like then $\overline{T}$-Cat $\cong \mathcal{G}V^T$.

Characterising the image of Γ

Note: if T is path-like then $\overline{T}\text{-Cat} \cong \mathcal{G}V^{T}$.

Theorem

Let V have coproducts. Then a monad T on $\mathcal{G}V$ over **Set** *is of the form $(\mathcal{G}V, \Gamma E)$ iff it is*

1. *distributive*
2. *path-like*

and in this case E is recaptured as $\overline{T}$.

Multitensors to operads

Given a cartesian monad T, recall that a T-operad is a cartesian monad morphism $\phi : A \to T$. Similarly given a cartesian multitensor E, one may define an E-multitensor to be a cartesian multitensor map into E.

Multitensors to operads

Given a cartesian monad T, recall that a T-operad is a cartesian monad morphism $\phi : A \to T$. Similarly given a cartesian multitensor E, one may define an E-multitensor to be a cartesian multitensor map into E.

Corollary

Let V be lextensive and E a cartesian multitensor on V. Then Γ and $\overline{(-)}$ induce

$$E\text{-multitensors} \simeq \Gamma E\text{-operads over } \mathbf{Set}.$$

Multitensors to operads

Given a cartesian monad T, recall that a T-operad is a cartesian monad morphism $\phi : A \to T$. Similarly given a cartesian multitensor E, one may define an E-multitensor to be a cartesian multitensor map into E.

Corollary

Let V be lextensive and E a cartesian multitensor on V. Then Γ and $\overline{(-)}$ induce

$$E\text{-multitensors} \simeq \Gamma E\text{-operads over } \mathbf{Set}.$$

Main case of interest: $E = \mathcal{T}^{\times}_{\leq n}$.

Contractibility

Let $\mathcal{I}$ be a class of maps in a category V. Recall that a morphism in V is a *trivial* $\mathcal{I}$*-fibration* when it satisfies RLP with respect to all elements of $\mathcal{I}$.

Contractibility

Let $\mathcal{I}$ be a class of maps in a category V. Recall that a morphism in V is a *trivial $\mathcal{I}$-fibration* when it satisfies RLP with respect to all elements of $\mathcal{I}$. We shall say that a natural transformation between functors into V is a trivial $\mathcal{I}$-fibration when each of its components is such.

Contractibility

Let $\mathcal{I}$ be a class of maps in a category V. Recall that a morphism in V is a *trivial $\mathcal{I}$-fibration* when it satisfies RLP with respect to all elements of $\mathcal{I}$. We shall say that a natural transformation between functors into V is a trivial $\mathcal{I}$-fibration when each of its components is such.

When V has an initial object any class $\mathcal{I}$ determines a class $\mathcal{I}^+$ of maps of $\mathcal{G}V$ containing

$$\emptyset \to () \qquad (i) : (S) \to (B)$$

where $i \in \mathcal{I}$.

Contractibility

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Starting with the empty class of maps in **Glob** iterating $(-)^+$ and taking the union, produces the class $\mathcal{I}_\infty$ containing the inclusion of the boundary of the free-living n-cell for each $n \in \mathbb{N}$.

Contractibility

Starting with the empty class of maps in **Glob** iterating $(-)^+$ and taking the union, produces the class $\mathcal{I}_\infty$ containing the inclusion of the boundary of the free-living n-cell for each $n \in \mathbb{N}$.

For the finite dimensional versions – classes in $\mathcal{G}^n$**Set** denoted $\mathcal{I}_{\leq n}$ – start with the class

$$\{\emptyset \to 1,\ 2 \to 1\}$$

in **Set** and successively apply $(-)^+$.

Contractibility

Starting with the empty class of maps in **Glob** iterating $(-)^+$ and taking the union, produces the class $\mathcal{I}_\infty$ containing the inclusion of the boundary of the free-living n-cell for each $n \in \mathbb{N}$.

For the finite dimensional versions – classes in $\mathcal{G}^n$**Set** denoted $\mathcal{I}_{\leq n}$ – start with the class

$$\{\emptyset \to 1,\, 2 \to 1\}$$

in **Set** and successively apply $(-)^+$.

Leinster: An n-operad is *contractible* when it is a trivial $\mathcal{I}_{\leq n}$-fibration.

Contractible multitensors and operads

Proposition

Let E be a $\mathcal{T}_{\leq n}$-multitensor. Then E is a contractible n-multitensor iff ΓE is a contractible $(n{+}1)$-operad.

Path space functor

There are two basic ingredients for the Trimble definition. The first is the path space functor

$$\mathbb{P} : \textbf{Top} \to \mathcal{G}\textbf{Top}$$

and this arises quite canonically in our setting.

Background from topology

For any space X the reduced suspension of X, is defined as the pushout

$$\begin{array}{ccc} X+X & \longrightarrow & I\times X \\ \downarrow & & \downarrow \\ 1+1 & \longrightarrow & \sigma X. \end{array}$$

and so we get

$$\mathbf{Top} \underset{h}{\overset{\sigma}{\rightleftarrows}} \mathbf{Top}_{\bullet}\quad (\sigma \dashv h)$$

where $h(a, X, b)$ is the space of paths in X from a to b.

Path space functor

On the categorical side one has the 2-adjunction

$$\mathbf{CAT}/\mathbf{Set} \underset{\mathcal{G}}{\overset{(-)_\bullet}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{CAT}$$

where for $f : A \to \mathbf{Set}$, $A_\bullet$ is the category of bipointed objects in A.

Path space functor

On the categorical side one has the 2-adjunction

$$\mathbf{CAT}/\mathbf{Set} \underset{\mathcal{G}}{\overset{(-)_\bullet}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{CAT}$$

where for $f : A \to \mathbf{Set}$, $A_\bullet$ is the category of bipointed objects in A.

The path space functor $\mathbb{P}$ is the composite

$$\mathbf{Top} \xrightarrow{\text{unit}} \mathcal{G}\mathbf{Top}_\bullet \xrightarrow{\mathcal{G}h} \mathcal{G}\mathbf{Top}$$

Path space functor

Introduction
Multitensors and Monads
Contractibility
Trimble and Cheng
Gray and Crans
Free Products
Relevant Properties

On the categorical side one has the 2-adjunction

$$\mathbf{CAT}/\mathbf{Set} \underset{\mathcal{G}}{\overset{(-)_\bullet}{\rightleftarrows}} \mathbf{CAT} \quad (\bot)$$

where for $f : A \to \mathbf{Set}$, $A_\bullet$ is the category of bipointed objects in A.

The path space functor $\mathbb{P}$ is the composite

$$\mathbf{Top} \xrightarrow{\text{unit}} \mathcal{G}\mathbf{Top}_\bullet \xrightarrow{\mathcal{G}h} \mathcal{G}\mathbf{Top}$$

and from this description $\mathbb{P}$ is evidently a right adjoint.

Operad action on path-spaces

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

The second basic ingredient for the Trimble definition is a non-symmetric contractible topological operad A which acts on the path space functor.

Operad action on path-spaces

The second basic ingredient for the Trimble definition is a non-symmetric contractible topological operad A which acts on the path space functor. To say that A acts on $\mathbb{P}$ is to say that $\mathbb{P}$ factors as

$$\mathbf{Top} \xrightarrow{P_A} A\text{-Cat} \xrightarrow{U^A} \mathcal{G}\mathbf{Top}$$

Operad action on path-spaces

The second basic ingredient for the Trimble definition is a non-symmetric contractible topological operad A which acts on the path space functor. To say that A acts on $\mathbb{P}$ is to say that $\mathbb{P}$ factors as

$$\mathbf{Top} \xrightarrow{P_A} A\text{-Cat} \xrightarrow{U^A} \mathcal{G}\mathbf{Top}$$

Since $\mathbb{P}$ is a right adjoint, P_A is also a right adjoint by the Dubuc adjoint triangle theorem.

Trimble's inductive machine

$$(Q, V) \mapsto (Q^{(+)}, V^{(+)})$$

Applies to pairs (Q, V) consisting of a distributive category V and a product preserving $Q : \textbf{Top} \to V$.

Trimble's inductive machine

$$(Q, V) \mapsto (Q^{(+)}, V^{(+)})$$

Applies to pairs (Q, V) consisting of a distributive category V and a product preserving $Q : \mathbf{Top} \to V$.

Regarding Q as a (strong) monoidal functor $(\mathbf{Top}, A) \to (V, QA)$, and applying Γ gives a monad morphism

$$(\mathcal{G}(\mathbf{Top}), \Gamma(A)) \to (\mathcal{G}V, \Gamma(QA))$$

with underlying functor $\mathcal{G}Q$.

Trimble's inductive machine

By formal monad theory this monad morphism amounts to giving a lifting $\overline{Q}$ as indicated in the commutative diagram

$$\begin{array}{ccccc}
\mathbf{Top} & \xrightarrow{P_A} & A\text{-Cat} & \xrightarrow{\overline{Q}} & QA\text{-Cat} \\
& \searrow^{P} & \downarrow & & \downarrow U^{QA} \\
& & \mathcal{G}(\mathbf{Top}) & \xrightarrow[\mathcal{G}(Q)]{} & \mathcal{G}V
\end{array}$$

Trimble's inductive machine

By formal monad theory this monad morphism amounts to giving a lifting $\overline{Q}$ as indicated in the commutative diagram

$$\begin{array}{ccccc} \mathbf{Top} & \xrightarrow{P_A} & A\text{-Cat} & \xrightarrow{\overline{Q}} & QA\text{-Cat} \\ & \searrow^{P} & \downarrow & & \downarrow U^{QA} \\ & & \mathcal{G}(\mathbf{Top}) & \xrightarrow[\mathcal{G}(Q)]{} & \mathcal{G}V \end{array}$$

and then one defines

$$Q^{(+)} = \overline{Q}P_A \qquad V^{(+)} = QA\text{-Cat}$$

Trimble's inductive machine

By formal monad theory this monad morphism amounts to giving a lifting $\overline{Q}$ as indicated in the commutative diagram

$$\begin{array}{ccccc} \mathbf{Top} & \xrightarrow{P_A} & A\text{-Cat} & \xrightarrow{\overline{Q}} & QA\text{-Cat} \\ & \searrow^{P} & \downarrow & & \downarrow U^{QA} \\ & & \mathcal{G}(\mathbf{Top}) & \xrightarrow[\mathcal{G}(Q)]{} & \mathcal{G}V \end{array}$$

and then one defines

$$Q^{(+)} = \overline{Q}P_A \qquad V^{(+)} = QA\text{-Cat}$$

Iterating this starting from $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ produces the fundamental n-groupoid functor from **Top** into the category of Trimble n-categories.

Cheng's theorem

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

Cheng's theorem

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

(*indication of proof*):

Cheng's theorem

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

(*indication of proof*):
The monads Trm_n are given inductively as

$$\mathrm{Trm}_0 = 1_{\mathbf{Set}} \qquad \mathrm{Trm}_{n+1} = \Gamma(\pi_n A)$$

by construction,

Cheng's theorem

Introduction
Multitensors and Monads
Contractibility
Trimble and Cheng
Gray and Crans
Free Products
Relevant Properties

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

(*indication of proof*):
The monads Trm_n are given inductively as

$$\mathrm{Trm}_0 = 1_{\mathbf{Set}} \qquad \mathrm{Trm}_{n+1} = \Gamma(\pi_n A)$$

by construction, and the cartesian multitensor map $A \to \prod$ can be used to construct the n-operad structure.

Cheng's theorem

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

(*indication of proof*):
The monads Trm_n are given inductively as

$$\mathrm{Trm}_0 = 1_{\mathbf{Set}} \qquad \mathrm{Trm}_{n+1} = \Gamma(\pi_n A)$$

by construction, and the cartesian multitensor map $A \to \prod$ can be used to construct the n-operad structure. Contractibility in the topological setting and contractibility in the globular setting are compatible because of how $\mathbb{P}$ arises,

Cheng's theorem

Theorem

(E. Cheng) Trimble n-categories are Batanin n-categories.

(*indication of proof*):
The monads Trm_n are given inductively as

$$\mathrm{Trm}_0 = 1_{\mathbf{Set}} \qquad \mathrm{Trm}_{n+1} = \Gamma(\pi_n A)$$

by construction, and the cartesian multitensor map $A \to \prod$ can be used to construct the n-operad structure. Contractibility in the topological setting and contractibility in the globular setting are compatible because of how $\mathbb{P}$ arises, and this together with the general theory enables one to verify the contractibility of Trm_n.

Recall that given an $(n+1)$-operad A over **Set**, $\overline{A}$ is an n-multitensor, so in particular gives a lax monoidal structure on $\mathcal{G}^n$**Set**, and $\overline{A}$-categories are the same as A-algebras.

Recall that given an $(n+1)$-operad A over **Set**, $\overline{A}$ is an n-multitensor, so in particular gives a lax monoidal structure on $\mathcal{G}^n\mathbf{Set}$, and $\overline{A}$-categories are the same as A-algebras.

Example

Let A be the 3-operad for Gray categories. Then $\overline{A}$ is a lax monoidal structure on 2-globular sets and $\overline{A}$-categories are Gray categories. Note that $\overline{A}_1$ is the monad for (strict) 2-categories.

Recall that given an $(n+1)$-operad A over **Set**, $\overline{A}$ is an n-multitensor, so in particular gives a lax monoidal structure on $\mathcal{G}^n\mathbf{Set}$, and $\overline{A}$-categories are the same as A-algebras.

Example

Let A be the 3-operad for Gray categories. Then $\overline{A}$ is a lax monoidal structure on 2-globular sets and $\overline{A}$-categories are Gray categories. Note that $\overline{A}_1$ is the monad for (strict) 2-categories.

Question: how are $\overline{A}$ and the Gray tensor product for 2-categories related?

Lemma

Let (V, E) be a lax monoidal category. Then (E_1, u, σ) is a monad and

1. *for any $(X_1, ..., X_n)$, $\underset{i}{E} X_i$ is an E_1-algebra.*
2. *for any E-category X, its homs are E_1-algebras.*

Lemma

Let (V, E) *be a lax monoidal category. Then* (E_1, u, σ) *is a monad and*

1. *for any* $(X_1, ..., X_n)$, $\underset{i}{E} X_i$ *is an* E_1*-algebra.*
2. *for any* E*-category* X*, its homs are* E_1*-algebras.*

Question: is there a canonical way to lift E to a multitensor E' on V^{E_1} so that E'-Cat $\cong$ E-Cat?

Lifting theorem

Theorem

Let E be an accessible distributive multitensor on a locally presentable category V. Then there is, to within isomorphism, a unique multitensor E' on V^{E_1} such that

1. *the unit for E' is the identity.*
2. *E' is distributive.*
3. E'-Cat $\cong$ E-Cat.

Lifting theorem

Indication of the proof: One has a commutative triangle of forgetful functors

$$\begin{array}{ccc} \mathcal{G}(V^{E_1}) & \longleftarrow & E\text{-Cat} \\ & \searrow \quad \swarrow & \\ & \mathcal{G}V & \end{array}$$

and so an induced monad T on $\mathcal{G}(V^{E_1})$.

Lifting theorem

Indication of the proof: One has a commutative triangle of forgetful functors

$$\begin{array}{ccc} \mathcal{G}(V^{E_1}) & \longleftarrow & E\text{-Cat} \\ & \searrow \quad \swarrow & \\ & \mathcal{G}V & \end{array}$$

and so an induced monad T on $\mathcal{G}(V^{E_1})$. This monad turns out to be distributive and path-like and so one can take $E' = \overline{T}$.

Lifting theorem

Indication of the proof: One has a commutative triangle of forgetful functors

$$\begin{array}{ccc} \mathcal{G}(V^{E_1}) & \longleftarrow & E\text{-Cat} \\ & \searrow \quad \swarrow & \\ & \mathcal{G}V & \end{array}$$

and so an induced monad T on $\mathcal{G}(V^{E_1})$. This monad turns out to be distributive and path-like and so one can take $E' = \overline{T}$.

For uniqueness let F be some multitensor with the desired properties.

Lifting theorem

Indication of the proof: One has a commutative triangle of forgetful functors

$$\begin{array}{ccc} \mathcal{G}(V^{E_1}) & \longleftarrow & E\text{-}\mathrm{Cat} \\ & \searrow \quad \swarrow & \\ & \mathcal{G}V & \end{array}$$

and so an induced monad T on $\mathcal{G}(V^{E_1})$. This monad turns out to be distributive and path-like and so one can take $E' = \overline{T}$.

For uniqueness let F be some multitensor with the desired properties. Then ΓF and T are monads on $\mathcal{G}(V^{E_1})$ with the same algebras,

Lifting theorem

Indication of the proof: One has a commutative triangle of forgetful functors

$$\begin{array}{ccc} \mathcal{G}(V^{E_1}) & \longleftarrow & E\text{-Cat} \\ & \searrow \qquad \swarrow & \\ & \mathcal{G}V & \end{array}$$

and so an induced monad T on $\mathcal{G}(V^{E_1})$. This monad turns out to be distributive and path-like and so one can take $E' = \overline{T}$.

For uniqueness let F be some multitensor with the desired properties. Then ΓF and T are monads on $\mathcal{G}(V^{E_1})$ with the same algebras, thus they're isomorphic and so $F \cong \overline{T}$.

Recall that if you take the Gray tensor product of 2-categories, and observe what happens in below dimension 2, that one is observing a canonical tensor product of categories.

Recall that if you take the Gray tensor product of 2-categories, and observe what happens in below dimension 2, that one is observing a canonical tensor product of categories.

Theorem

(Flotz, Kelly and Lair) Up to isomorphism there are exactly two biclosed monoidal structures on **Cat***, both symmetric.*

Recall that if you take the Gray tensor product of 2-categories, and observe what happens in below dimension 2, that one is observing a canonical tensor product of categories.

Theorem

(Flotz, Kelly and Lair) Up to isomorphism there are exactly two biclosed monoidal structures on **Cat**, *both symmetric.*

The "other one" is often called the "funny" tensor product, but we call it and its generalisations *free products*.

Basics on the free product of categories

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

- there's an identity on objects comparison $A \otimes B \longrightarrow A \times B$ natural in A and B.

Basics on the free product of categories

- there's an identity on objects comparison $A \otimes B \to A \times B$ natural in A and B.
- one has the *pushout formula*:

$$\begin{array}{ccc} A_0 \times B_0 & \xrightarrow{\text{id} \times i_B} & A_0 \times B \\ {\scriptstyle i_A \times \text{id}} \downarrow & & \vdots \\ A \times B_0 & \cdots\cdots\to & A \otimes B \end{array}$$

Basics on the free product of categories

- there's an identity on objects comparison $A \otimes B \to A \times B$ natural in A and B.
- one has the *pushout formula*:

$$\begin{array}{ccc} A_0 \times B_0 & \xrightarrow{\mathrm{id} \times i_B} & A_0 \times B \\ {\scriptstyle i_A \times \mathrm{id}} \downarrow & & \vdots \\ A \times B_0 & \cdots\cdots\to & A \otimes B \end{array}$$

- explicitly, morphisms of $A \otimes B$ are generated by

$$(a, \beta) : (a, b_1) \to (a, b_2) \qquad (\alpha, b) : (a_1, b) \to (a_2, b)$$

subject to relations remembering composition in A and B.

Basics on the free product of categories

- there's an identity on objects comparison $A \otimes B \to A \times B$ natural in A and B.
- one has the *pushout formula*:

$$\begin{array}{ccc} A_0 \times B_0 & \xrightarrow{\mathrm{id} \times i_B} & A_0 \times B \\ {\scriptstyle i_A \times \mathrm{id}} \downarrow & & \vdots \\ A \times B_0 & \cdots\cdots\to & A \otimes B \end{array}$$

- explicitly, morphisms of $A \otimes B$ are generated by

$$(a, \beta) : (a, b_1) \to (a, b_2) \qquad (\alpha, b) : (a_1, b) \to (a_2, b)$$

subject to relations remembering composition in A and B.
- $[A, B]$ consists of functors and "transformations".

Basics on the free product of categories

For maps α and β the square

$$\begin{array}{ccc} (a_1, b_1) & \xrightarrow{(a_1,\beta)} & (a_1, b_2) \\ {\scriptstyle (\alpha,b_1)}\downarrow & & \downarrow{\scriptstyle (\alpha,b_2)} \\ (a_2, b_1) & \xrightarrow[(a_2,\beta)]{} & (a_2, b_2) \end{array}$$

commutes in the cartesian product, but **not** in the free product.

Basics on the free product of categories

For maps α and β the square

$$\begin{array}{ccc} (a_1, b_1) & \xrightarrow{(a_1,\beta)} & (a_1, b_2) \\ {\scriptstyle (\alpha,b_1)}\downarrow & & \downarrow{\scriptstyle (\alpha,b_2)} \\ (a_2, b_1) & \xrightarrow[(a_2,\beta)]{} & (a_2, b_2) \end{array}$$

commutes in the cartesian product, but **not** in the free product.

The Gray tensor product proceeds in the same way for objects and arrows, and in dimension 2 one has a coherent isomorphism between the two diagonals of this square.

For any n-operad over **Set** there is an analogue of the free product for its algebras.

Main ideas for the generalised free products

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathrm{SMltCAT} \xleftarrow{\mathcal{U}} \mathrm{SMonCAT}$$

Main ideas for the generalised free products

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \text{SMltCAT} \xleftarrow{\mathcal{U}} \text{SMonCAT}$$

$(\mathcal{G}V, T)$

Main ideas for the generalised free products

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathrm{SMltCAT} \xleftarrow{\mathcal{U}} \mathrm{SMonCAT}$$

$$(\mathcal{G}V, T) \longmapsto (\mathcal{F}\mathcal{G}V, \mathcal{F}T)$$

To be described: $\mathcal{F}$ is an EM-object-preserving 2-functor over **CAT**.

Main ideas for the generalised free products

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \text{SMltCAT} \xleftarrow{\mathcal{U}} \text{SMonCAT}$$

$$(\mathcal{G}V, T) \longmapsto (\mathcal{F}\mathcal{G}V, \mathcal{F}T) \leftarrow\!\!\cdots\!\!\shortmid ((\mathcal{G}V, \otimes), T)$$

To be described: $\mathcal{F}$ is an EM-object-preserving 2-functor over **CAT**.

Hermida: $\mathcal{U}$ is 2-fully-faithful and its image consists of the representable multicategories.

Main ideas for the generalised free products

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathrm{SMltCAT} \xleftarrow{\mathcal{U}} \mathrm{SMonCAT}$$

$$(\mathcal{G}V, T) \longmapsto (\mathcal{F}\mathcal{G}V, \mathcal{F}T) \dashleftarrow ((\mathcal{G}V, \otimes), T)$$

To be described: $\mathcal{F}$ is an EM-object-preserving 2-functor over **CAT**.

Hermida: $\mathcal{U}$ is 2-fully-faithful and its image consists of the representable multicategories. Under mild conditions on V the multicategory $\mathcal{F}\mathcal{G}V$ is representable and closed (in the sense of Manzyuk).

Main ideas for the generalised free products

$$\mathbf{CAT}/\mathbf{Set} \xrightarrow{\mathcal{F}} \mathrm{SMltCAT} \xleftarrow{\mathcal{U}} \mathrm{SMonCAT}$$

$$(\mathcal{G}V, T) \longmapsto (\mathcal{F}\mathcal{G}V, \mathcal{F}T) \leftarrow\!\!\!\dashv ((\mathcal{G}V, \otimes), T)$$

To be described: $\mathcal{F}$ is an EM-object-preserving 2-functor over **CAT**.

Hermida: $\mathcal{U}$ is 2-fully-faithful and its image consists of the representable multicategories. Under mild conditions on V the multicategory $\mathcal{F}\mathcal{G}V$ is representable and closed (in the sense of Manzyuk). By the 2-fully-faithfulness of $\mathcal{U}$, T is automatically a symmetric monoidal monad.

Symmetric monoidal monads

Introduction
Multitensors and Monads
Contractibility
Trimble and Cheng
Gray and Crans
Free Products
Relevant Properties

Theorem

(A. Kock) Let T be a symmetric monoidal monad on V a symmetric monoidal closed category with equalisers. Then the equaliser

$$[(X,x),(Y,y)] \dashrightarrow [X,Y] \xrightarrow{[x,\mathrm{id}]} [TX,Y], \qquad [X,Y] \to [TX,TY] \xrightarrow{[\mathrm{id},y]} [TX,Y]$$

defines the internal hom and (TI,μ_I) the unit of a closed structure on V^T.

Symmetric monoidal monads

Theorem

(F. Linton, B. Day) If in addition V^T has coequalisers then the coequaliser

$$
\begin{array}{ccccc}
T \bigotimes\limits_i TX_i & \xrightarrow{\;T \bigotimes\limits_i x_i\;} & T \bigotimes\limits_i X_i & \dashrightarrow & \bigotimes\limits_i (X_i, x_i) \\
& \searrow & \nearrow_{\mu} & & \\
& & T^2 \bigotimes\limits_i X_i & &
\end{array}
$$

defines the associated tensor product on V^T making it symmetric monoidal closed.

Definition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. The tensor product on $\mathcal{G}V^T$ is called the *free product of T-algebras*.

Description of $\mathcal{F}$

Let A be a category over **Set** and for $a \in A$, denote by a_0 the underlying set of a.

Description of $\mathcal{F}$

Let A be a category over **Set** and for $a \in A$, denote by a_0 the underlying set of a. A multimap of $\mathcal{F}A$

$$f : (a_1, ..., a_n) \longrightarrow b$$

consists of a function

$$f_0 : a_{1,0} \times ... \times a_{n,0} \longrightarrow b_0$$

Description of $\mathcal{F}$

Let A be a category over **Set** and for $a \in A$, denote by a_0 the underlying set of a. A multimap of $\mathcal{F}A$

$$f : (a_1, ..., a_n) \longrightarrow b$$

consists of a function

$$f_0 : a_{1,0} \times ... \times a_{n,0} \longrightarrow b_0$$

together with for each $1 \leq i^* \leq n$ and $z \in \prod_{i \neq i^*} a_{i,0}$, a morphism $f_z : a_{i^*} \longrightarrow b$ of A such that $(f_z)_0 = (f_0)_z$.

Description of $\mathcal{F}$

Let A be a category over **Set** and for $a \in A$, denote by a_0 the underlying set of a. A multimap of $\mathcal{F}A$

$$f : (a_1, ..., a_n) \to b$$

consists of a function

$$f_0 : a_{1,0} \times ... \times a_{n,0} \to b_0$$

together with for each $1 \leq i^* \leq n$ and $z \in \prod\limits_{i \neq i^*} a_{i,0}$, a morphism $f_z : a_{i^*} \to b$ of A such that $(f_z)_0 = (f_0)_z$.

A nullary multimap is just an object of b and a linear map is just a morphism of A.

Description of $\mathcal{F}$

Example

$\mathcal{F}R$-Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Description of $\mathcal{F}$

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Example

$\mathcal{F}R$-Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Example

Let X be a set and M the monoid of endofunctions of X.

Description of $\mathcal{F}$

Example

$\mathcal{F}R$-Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Example

Let X be a set and M the monoid of endofunctions of X. The monoid M acts on X by evaluation giving a functor $M \to \mathbf{Set}$.

Description of $\mathcal{F}$

Example

$\mathcal{F}R$-Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Example

Let X be a set and M the monoid of endofunctions of X. The monoid M acts on X by evaluation giving a functor $M \to \mathbf{Set}$. The symmetric multicategory $\mathcal{F}M$ has one object, thus it is just an operad of sets.

Description of $\mathcal{F}$

Example

$\mathcal{F}R$-Mod is the symmetric multicategory of R-modules and R-multilinear maps.

Example

Let X be a set and M the monoid of endofunctions of X. The monoid M acts on X by evaluation giving a functor $M \rightarrow \mathbf{Set}$. The symmetric multicategory $\mathcal{F}M$ has one object, thus it is just an operad of sets. In fact $\mathcal{F}M$ is the endomorphism operad of X.

Canonical map: free product $\longrightarrow$ cartesian product

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Bourke: the Gray tensor product for 2-categories can be obtained by factoring the canonical map from the free to the cartesian product.

Canonical map: free product $\longrightarrow$ cartesian product

Bourke: the Gray tensor product for 2-categories can be obtained by factoring the canonical map from the free to the cartesian product.

There is no such map for mere graphs. So one can ask: when is there such a canonical map?

Canonical map: free product $\longrightarrow$ cartesian product

The unit for the free product on $\mathcal{G}V$ is denoted 0: the V-graph with one object and initial hom. Given a monad T on $\mathcal{G}V$, the unit for the free product on $\mathcal{G}V^T$ is $(T0, \mu_0)$.

Canonical map: free product $\to$ cartesian product

Introduction
Multitensors and Monads
Contractibility
Trimble and Cheng
Gray and Crans
Free Products
Relevant Properties

The unit for the free product on $\mathcal{G}V$ is denoted 0: the V-graph with one object and initial hom. Given a monad T on $\mathcal{G}V$, the unit for the free product on $\mathcal{G}V^T$ is $(T0, \mu_0)$.

If we had

$$A \otimes B \to A \times B$$

then putting $A = 1$ and $B = (T0, \mu_0)$ would give a map

$$e : 1 \to (T0, \mu_0).$$

Canonical map: free product $\rightarrow$ cartesian product

The unit for the free product on $\mathcal{G}V$ is denoted 0: the V-graph with one object and initial hom. Given a monad T on $\mathcal{G}V$, the unit for the free product on $\mathcal{G}V^T$ is $(T0, \mu_0)$.

If we had

$$A \otimes B \rightarrow A \times B$$

then putting $A = 1$ and $B = (T0, \mu_0)$ would give a map

$$e : 1 \rightarrow (T0, \mu_0).$$

It is obviously a split monomorphism

Canonical map: free product $\longrightarrow$ cartesian product

The unit for the free product on $\mathcal{G}V$ is denoted 0: the V-graph with one object and initial hom. Given a monad T on $\mathcal{G}V$, the unit for the free product on $\mathcal{G}V^T$ is $(T0, \mu_0)$.

If we had

$$A \otimes B \longrightarrow A \times B$$

then putting $A = 1$ and $B = (T0, \mu_0)$ would give a map

$$e : 1 \longrightarrow (T0, \mu_0).$$

It is obviously a split monomorphism. In fact it's an isomorphism

Canonical map: free product $\to$ cartesian product

The unit for the free product on $\mathcal{G}V$ is denoted 0: the V-graph with one object and initial hom. Given a monad T on $\mathcal{G}V$, the unit for the free product on $\mathcal{G}V^T$ is $(T0, \mu_0)$.

If we had

$$A \otimes B \to A \times B$$

then putting $A = 1$ and $B = (T0, \mu_0)$ would give a map

$$e : 1 \to (T0, \mu_0).$$

It is obviously a split monomorphism. In fact it's an isomorphism: to see that the composite

$$T0 \longrightarrow 1 \xrightarrow{e} T0$$

is the identity it suffices that the composite morphism

$$0 \xrightarrow{\eta_0} T0 \longrightarrow 1 \xrightarrow{e} T0$$

is η_0.

Canonical map: free product $\longrightarrow$ cartesian product

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

Proposition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. There is an identity on objects morphism $\otimes \longrightarrow \times$ of tensor products iff $T0 = 1$.

Canonical map: free product $\longrightarrow$ cartesian product

Proposition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. There is an identity on objects morphism $\otimes \longrightarrow \times$ of tensor products iff $T0 = 1$.

Such monads T are said to be *well-pointed*.

Canonical map: free product $\longrightarrow$ cartesian product

Proposition

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible. There is an identity on objects morphism $\otimes \longrightarrow \times$ of tensor products iff $T0 = 1$.

Such monads T are said to be *well-pointed*.

When T is coproduct preserving and satisfies the conditions of the proposition, the free product of T-algebras satisfies the pushout formula.

Monads and operads for sesqui-algebras

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

A *sesqui-T-algebra* is a category enriched in $\mathcal{G}V^T$ for the free product.

Monads and operads for sesqui-algebras

A *sesqui-T-algebra* is a category enriched in $\mathcal{G}V^T$ for the free product.

Question: if T is an n-operad, then are sesqui-T-algebras $(n+1)$-operadic?

Monads and operads for sesqui-algebras

Introduction

Multitensors and Monads

Contractibility

Trimble and Cheng

Gray and Crans

Free Products

Relevant Properties

A *sesqui-T-algebra* is a category enriched in $\mathcal{G}V^T$ for the free product.

Question: if T is an n-operad, then are sesqui-T-algebras $(n+1)$-operadic?

Theorem

Let V be locally presentable and the monad T on $\mathcal{G}V$ be accessible and coproduct preserving. Then the monad on $\mathcal{G}^2V$ whose algebras are sesqui-T-algebras is given explicitly as $\Gamma(T\bigotimes)$.

Theorem

Let V be locally presentable and extensive, and T be a monad on $\mathcal{G}V$ over **Set** *which is accessible, well-pointed, l.r.a, distributive and path-like. Suppose that $\psi : A \to T$ is a T-operad. Then*

$$A \bigotimes_i X_i \xrightarrow{\psi_{\bigotimes_i} X_i} T \bigotimes_i X_i \longrightarrow \prod_i TX_i$$

are the components of a $T^{\times}$-multitensor.