2-bicategories of polynomials

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Background on lccc's

Recall given $f : X \to Y$ in $\mathcal{E}$ a category with pullbacks

$$\mathcal{E}/X \; \begin{array}{c} \xrightarrow{\quad\Sigma_f\quad} \\ \perp \\ \xleftarrow{\quad\Delta_f\quad} \\ \perp \\ \xrightarrow[\quad\Pi_f\quad]{} \end{array} \; \mathcal{E}/Y$$

When Δ_f has a further right adjoint, denoted Π_f, f is said to be **exponentiable**.

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$$\mathcal{E}/X \underset{\Pi_f}{\overset{\Sigma_f}{\rightleftarrows}} \mathcal{E}/Y \qquad \Sigma_f \dashv \Delta_f \dashv \Pi_f, \quad \Delta_f : \mathcal{E}/Y \to \mathcal{E}/X$$

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When Δ_f has a further right adjoint, denoted Π_f, f is said to be **exponentiable**. When $\mathcal{E}$ has finite limits and all its morphisms are exponentiable, $\mathcal{E}$ is said to be **locally cartesian closed**. Toposes are l.c.c but **CAT** is not.

Key notions

The **polynomial functor** $\mathcal{E}/X \to \mathcal{E}/Y$ associated to a polynomial

$$p \quad : \quad X \xleftarrow{p_1} A \xrightarrow{p_2} B \xrightarrow{p_3} Y$$

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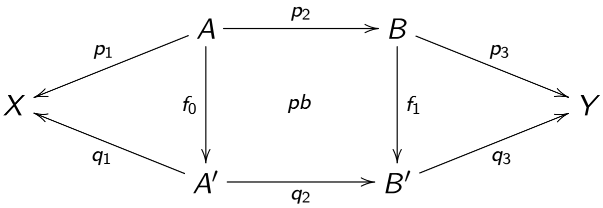
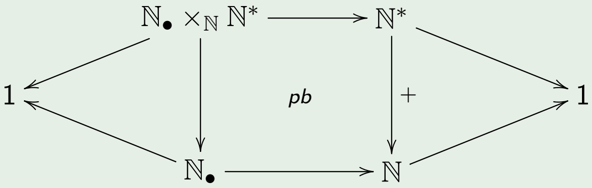
is the composite $\mathbf{P}(p) := \Sigma_{p_3}\Pi_{p_2}\Delta_{p_1}$. A morphism of polynomials is a diagram of the form



and induces a cartesian transformation $\mathbf{P}(p) \to \mathbf{P}(q)$.

Monoid monad on **Set**

Example



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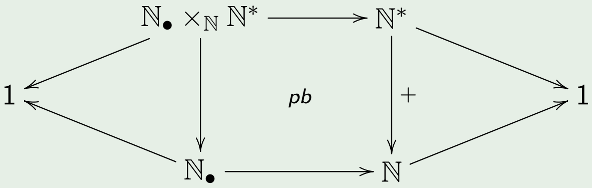

gives rise to the multiplication for the monoid monad.

Theorem

(Gambino and Kock 2009) Let $\mathcal{E}$ be locally cartesian closed.

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However the examples of polynomials we are interested in are in **CAT**. Also of interest are polynomials in **Top** – Bisson and Joyal, *The Dyer-Lashof Algebra in Bordism*, 1995.

Outline

1. Basic generalisation

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2. Enrichment over $\mathbf{CAT}_{\text{pb}}$ and iterability
3. The 2-bicategory of polynomials in a 2-category
4. Examples from 2-category theory

Outline

Generalisation to all categories with pullbacks

Theorem

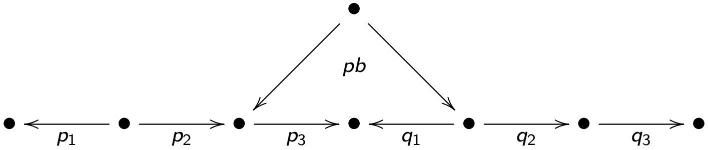
Let $\mathcal{E}$ be a category with pullbacks.

1. *Objects of $\mathcal{E}$, polynomials over $\mathcal{E}$ and morphisms of polynomials form a bicategory $\mathbf{Poly}_{\mathcal{E}}$.*
2. *The construction of polynomial functors from polynomials gives a homomorphism $\mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \to \mathbf{CAT}$.*

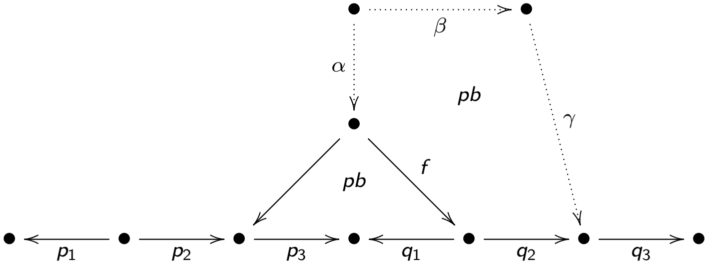
Composition of polynomials

$$\bullet \xleftarrow[p_1]{} \bullet \xrightarrow[p_2]{} \bullet \xrightarrow[p_3]{} \bullet \xleftarrow[q_1]{} \bullet \xrightarrow[q_2]{} \bullet \xrightarrow[q_3]{} \bullet$$

Composition of polynomials



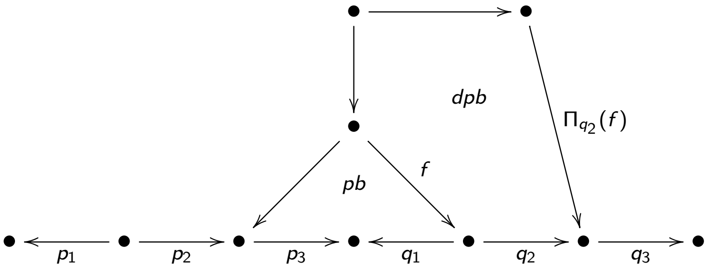
At this point one can consider the category of triples of morphisms (α, β, γ) as shown



making the square with boundary $(f\alpha, q_2, \gamma, \beta)$ a pullback.

Composition of polynomials

The terminal such is the **distributivity pullback** of f along q_2.



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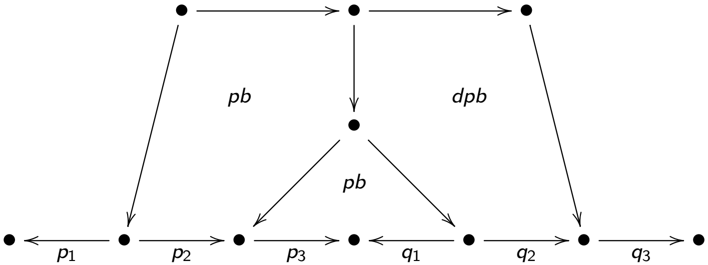

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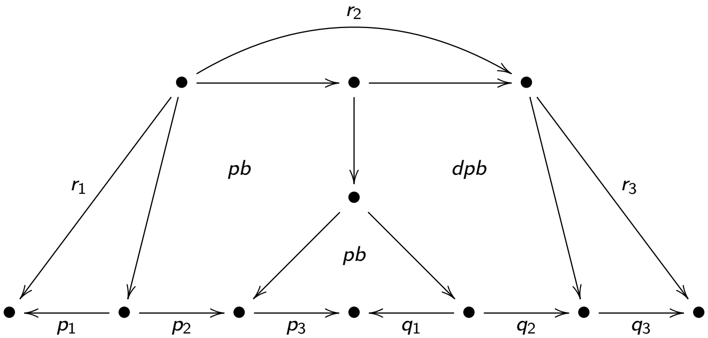


Lemma

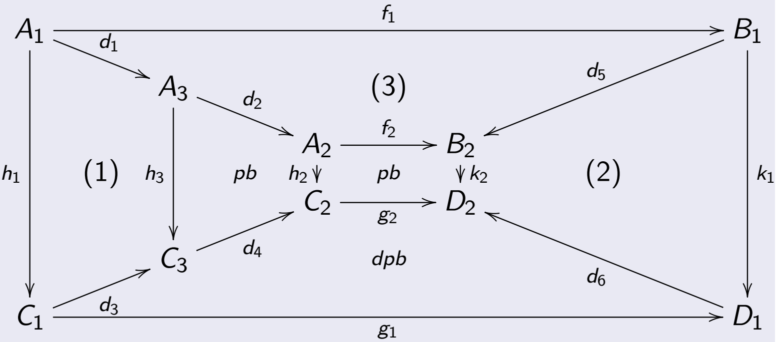
(Composition/cancellation) Given

$$
\begin{array}{ccccc}
B_6 & \xrightarrow{h_9} & B_4 & \xrightarrow{h_6} & B_5 \\
{\scriptstyle h_8}\downarrow & pb & \downarrow{\scriptstyle h_5} & & \\
B_2 & \xrightarrow[h_3]{} & B_3 & & \\
{\scriptstyle h_2}\downarrow & & & pb & \downarrow{\scriptstyle h_7} \\
B & dpb & \downarrow{\scriptstyle h_4} & & \\
{\scriptstyle h}\downarrow & & & & \\
X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z
\end{array}
$$

in any category with pullbacks, then the right-most pullback is a distributivity pullback around (g, h_4) *iff the composite diagram is a distributivity pullback around* (gf, h)*.*

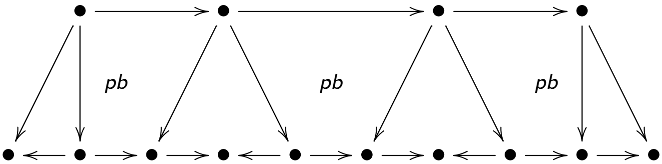
Lemma

(The cube lemma). Given



where (3) is a pullback. Then (1) and (2) are pullbacks iff (3) is a distributivity pullback.

Unbiased composition of polynomials



Enrichment over $\mathbf{CAT}_{\text{pb}}$

The category $\mathbf{CAT}_{\text{pb}}$ of categories with pullbacks and pullback preserving functors is cartesian closed. The internal hom $[X, Y]$ is the category of pullback preserving functors $X \to Y$ and cartesian transformations between them.

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A $\mathbf{CAT}_{\mathrm{pb}}$-*bicategory* is a bicategory $\mathcal{B}$ whose homs have pullbacks and whose compositions

$$\mathrm{comp}_{X,Y,Z} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$$

preserve them. Categories enriched in $\mathbf{CAT}_{\mathrm{pb}}$ are exactly those $\mathbf{CAT}_{\mathrm{pb}}$-bicategories whose underlying bicategory is a 2-category.

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A *homomorphism* $F : \mathcal{B} \to \mathcal{C}$ of $\mathbf{CAT}_{\text{pb}}$-bicategories is a homomorphism of their underlying bicategories whose hom functors preserve pullbacks.

Iterability of the theory

Theorem

Let $\mathcal{E}$ be a category with pullbacks. Then $\mathbf{Poly}_{\mathcal{E}}$ *is a* $\mathbf{CAT}_{\mathrm{pb}}$*-bicategory and*

$$\mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \to \mathbf{CAT}_{\mathrm{pb}}$$

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is a homomorphism of $\mathbf{CAT}_{\mathrm{pb}}$*-bicategories.*

Since the homs of $\mathbf{Poly}_{\mathcal{E}}$ also have pullbacks we can apply the theorem to any of those homs in place of $\mathcal{E}$.

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$$\text{3-categories} \quad \subset \quad \text{2-bicategories} \quad \subset \quad \text{Tricategories}$$

Theorem

Let $\mathcal{K}$ be a 2-category with pullbacks. Then $\mathbf{Poly}_{\mathcal{K}}$ *is a 2-bicategory and*

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A **pseudo-monad** on an object X of a 2-bicategory $\mathcal{B}$, is a pseudo-monoid in the monoidal 2-category $\mathcal{B}(X, X)$.

2-topos examples

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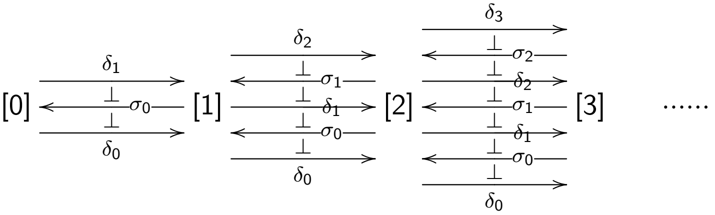
More generally, replace **CAT** by a finitely complete $\mathcal{K}$ whose discrete opfibrations are exponentiable and U by a classifying discrete opfibration in $\mathcal{K}$.

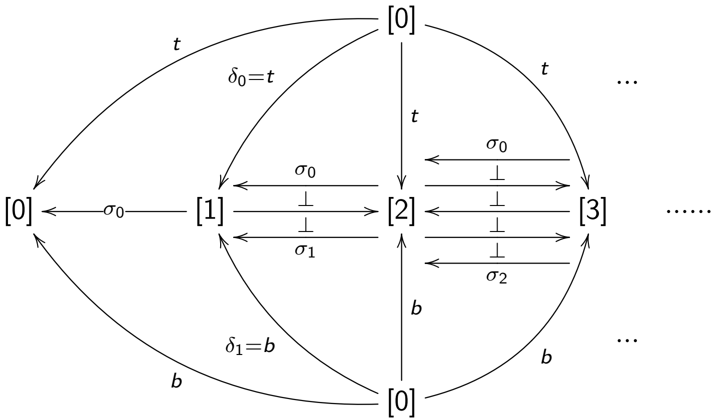
Street's internal fibrations

Recall that the inclusion $\Delta \hookrightarrow \mathbf{CAT}$ is a cocategory object, and its canonical generators enjoy some lovely adjointnesses



Street's internal fibrations

Reinterpretting a little ...



is a lax idempotent pseudo monad (on [1]) in the 2-bicategory $\mathbf{Cospan}_{\mathbf{CAT}}$.

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The associated pseudo monad on $\mathcal{K}/X$ is the monad for fibrations.

Local right adjoints vs polynomial functors

A functor $F : \mathcal{A} \to \mathcal{B}$ is a **local right adjoint** when for all $X \in \mathcal{A}$ the induced functor

$$F_X : \mathcal{A}/X \to \mathcal{B}/FX$$

is a right adjoint. When $\mathcal{A}$ has 1, it suffices to check this for $X = 1$.

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Polynomial functors are l.r.a because for a polynomial p, the composite $\Pi_{p_2}\Delta_{p_1}$ may be identified with $\mathbf{P}_{\mathcal{E}}(p)_1$. Notice that the left adjoint to $\mathbf{P}_{\mathcal{E}}(p)_1$ is $\Sigma_{p_1}\Delta_{p_2}$ which itself preserves connected limits and thus in particular monos.

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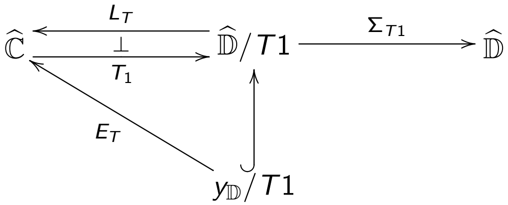
Given *Polynomial functors and opetopes* – BJKM 2007, this is a little sad.

Polynomial from a l.r.a between presheaf categories

Given $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ l.r.a one has



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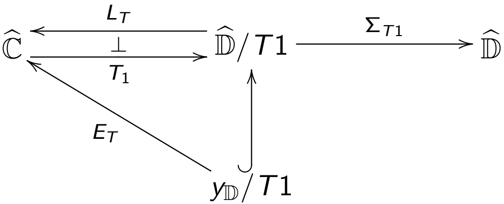

From which one produces the polynomial $p_T : \mathbb{C} \to \mathbb{D}$

$$\mathbb{C} \xleftarrow{p_{T,1}} y_{\mathbb{C}}/E_T \xrightarrow{p_{T,2}} y_{\mathbb{D}}/T1 \xrightarrow{p_{T,3}} \mathbb{D}$$

Proposition

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1. *Directly as* $T \cong \mathrm{lan}_{p_3}\mathrm{ran}_{p_2}\mathrm{res}_{p_1}$.
2. *By applying* $\mathbf{P}(p_T)$ *to discrete fibrations.*