Structured colimits and polynomial functors

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joint with
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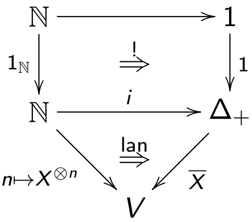
Free Monoids

If V is a distributive monoidal category then the formula

$$\overline{X}(1) = \coprod_{n \in \mathbb{N}} X^{\otimes n}$$

describing the free monoid on X arises by left extending along the inclusion of objects i:



This presentation of the free monoid construction depends on

1. The universal property of $\mathbb{N}$: $\text{Ps-}M\text{-Alg}(\mathbb{N}, V) \simeq V$.
2. That of Δ_+: $\text{Ps-}M\text{-Alg}(\Delta_+, V) \simeq \text{Mon}(V)$.
3. The distributivity of V.
4. Left kan extending a strong monoidal functor along i produces a strong monoidal functor.

In this talk we present a setting in which (1)-(3) and the functor i are definable from a monad theoretic setting, and (4) is explained via general results.

The unit η can be thought of as

- The 2-cell datum for a lax morphism of 2-monads $(1, \eta) : (\mathbf{Cat}, M) \to (\mathbf{Cat}, 1_{\mathbf{Cat}})$.
- The 2-cell datum for an oplax morphism of 2-monads $(1, \eta) : (\mathbf{Cat}, 1_{\mathbf{Cat}}) \to (\mathbf{Cat}, M)$.

This situation is summarised in the picture

$$(\mathbf{Cat}, M) \underset{(1,\eta)}{\overset{(1,\eta)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathbf{Cat}, 1)$$

General setting

The general setting involves three 2-monads and monad morphisms

$$(\mathcal{K}, T) \underset{(F,\phi)}{\overset{(\tilde{F},\tilde{\phi})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathcal{L}, S) \underset{(G,\gamma)}{\overset{(\tilde{G},\tilde{\gamma})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathcal{M}, R)$$

which play the following roles.

- T: the structure of the environment (monoidal category).
- S: the richer structure considered within T-algebras (monoids in monoidal categories).
- R: the simpler structure considered within T-algebras (objects in monoidal categories).
- (F, ϕ) and $(H, \psi) = (G, \gamma)(F, \phi)$ is data necessary to define what S and R-algebras within T-algebras are.

Recovering the ingredients of the free monoid construction

Monoids in V can be thought of as

- Lax M-morphisms $1 \to V$, by definition.
- When V is strict, as strict M-algebra morphisms $\Delta_+ \to V$.
- For general V, as strong M-algebra morphisms $\Delta_+ \to V$.

Objects of V can be thought of as

- Lax $1_{\mathbf{Cat}}$-morphisms $1 \to V$, by definition.
- When V is strict, as strict M-algebra morphisms $\mathbb{N} \to V$.
- For general V, as strong M-algebra morphisms $\mathbb{N} \to V$.

Obtaining $\mathbb{N}$, Δ_+ and i from the general setting

Given

$$(\mathcal{K}, T) \underset{(F,\phi)}{\overset{(\tilde{F},\tilde{\phi})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathcal{L}, S)$$

the 2-functor

$$T\text{-}\mathrm{Alg_s} \to S\text{-}\mathrm{Alg_l} \qquad TA \xrightarrow{a} A \mapsto SFA \xrightarrow{\phi_A} FTA \xrightarrow{Fa} FA$$

has a left adjoint when T-$\mathrm{Alg_s}$ admits codescent objects. The effect on 1 of this left adjoint is denoted T^S.

Thus by definition T^S is universal among strict T-algebras

$$T\text{-}\mathrm{Alg}_{\mathrm{s}}(T^S, V) \cong S\text{-}\mathrm{Alg}_{\mathrm{l}}(1, FV)$$

and when $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ complete and cocomplete, and T preserves bijections on objects, T^S is also universal among pseudo-T-algebras

$$\mathrm{Ps}\text{-}T\text{-}\mathrm{Alg}(T^S, V) \simeq \mathrm{Ps}\text{-}S\text{-}\mathrm{Alg}_{\mathrm{l}}(1, FV)$$

since it's flexible and using Power's coherence theorem for T-algebras.

Reformulating $i : \mathbb{N} \to \Delta_+$

Applying F to the universal map $1 \to T^T$ and applying T^S's universal property gives

$$T^{\phi} : T^S \to T^T$$

which is $i : \mathbb{N} \to \Delta_+$ when

$$(F, \phi) = (1, \eta) : (\mathbf{Cat}, M) \to (\mathbf{Cat}, 1).$$

Reminder of codescent objects

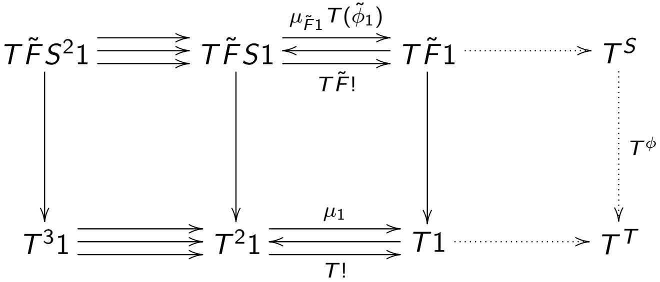
The colimit of $X : \Delta^{\mathrm{op}} \to \mathcal{K}$ weighted by $\Delta \hookrightarrow$ **Cat** is called the **codescent object** of X. The universal cocone of this colimit consists of $(q, \overline{q})$

$$X_2 \rightrightarrows\!\!\!\to X_1 \underset{d_0}{\overset{d_1}{\rightleftarrows\!\!\!\to}} X_0 \xrightarrow{q} \mathrm{CD}(X)$$

where $\overline{q} : qd_1 \to qd_0$ such that

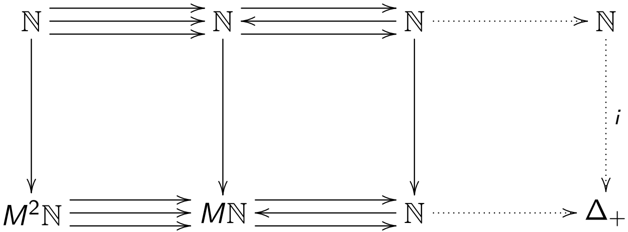
$$\overline{q}s_0 = \mathrm{id} \qquad (\overline{q}d_0)(\overline{q}d_2) = \overline{q}d_1.$$

T^ϕ in terms of codescent objects



When T preserves codescent objects, these codescent objects may be computed in $\mathcal{K}$. This enables one to unpack the square witnessing T^ϕ is a strict T-algebra map.

$i : \mathbb{N} \to \Delta_+$ in terms of codescent objects



The assignation

$$(\mathcal{K}, T) \xrightarrow{(F,\phi)} (\mathcal{L}, S) \mapsto T^S \xrightarrow{T^\phi} T^T$$

is object map of a functor, the arrow map of which produces

$$\begin{array}{ccc} T^R & \xrightarrow{T^\gamma} & T^S \\ & \searrow \quad = \quad \swarrow & \\ & T^T & \end{array}$$

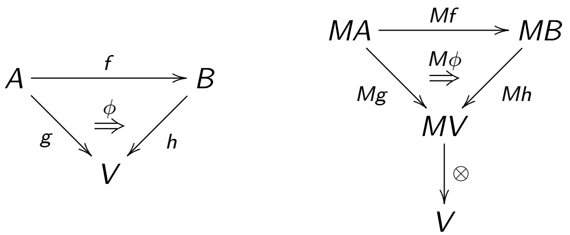
from the general setting.

Distributivity of V. One can say that V has coproducts by saying it admits pointwise left kan extensions along the following classes $\mathcal{C}$ of functors

- $f : A \to B$ where A and B are small discrete.
- $f : A \to B$ where A is small discrete and B's endomorphisms are all identities.

In general we say that the object V of **Cat** is **cocomplete** relative to all $f \in \mathcal{C}$.

The preservation in each variable of $\coprod$'s by V's $\otimes$ can be formalised by saying that if ϕ is a pointwise left extension



where $f \in \mathcal{C}$, then so is the diagram on the right.

In general we say that the pseudo M-algebra V is **algebraically cocomplete** relative to all $f \in \mathcal{C}$.

General idea: left kan extension along $T^{\gamma} : T^{R} \to T^{S}$ is some kind of meaningful construction, the combinatorics of which is encapsulated by the monads and monad morphisms from which it arose.

Questions:

1. Find general conditions on strict T-morphisms so that left extending a strong T-morphism along it is a strong T-morphism.
2. Sufficient conditions on the monad morphisms so that T^{γ} satisfies (1).

Exactness of a square

$$\begin{array}{ccc} D & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & \overset{\psi}{\Longrightarrow} & \downarrow{\scriptstyle x} \\ A & \xrightarrow[f]{} & B \end{array}$$

in a 2-category $\mathcal{K}$ with comma objects: if ϕ is p.l.e then so is its composite with ψ.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \overset{\phi}{\Longrightarrow} & \\ {\scriptstyle g}\searrow & & \swarrow{\scriptstyle h} \\ & X & \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & \overset{\psi}{\Longrightarrow} & \downarrow{\scriptstyle x} \\ A & \xrightarrow{f} & B \\ & \overset{\phi}{\Longrightarrow} & \\ {\scriptstyle g}\searrow & & \swarrow{\scriptstyle h} \\ & X & \end{array}$$

Proposition

In any 2-category with comma objects the following squares are exact:

- *Comma squares.*
- *Pullback squares in which f is an opfibration.*
- *Pullback squares in which x is a fibration.*
- *Bipullback squares in which f is a bi-opfibration.*
- *Bipullback squares in which x is a bi-fibration.*

A 2-functor $T : \mathcal{A} \to \mathcal{B}$ between 2-categories with finite limits is **opfamilial** when

- $T_1 : \mathcal{A} \to B/T1$ has a left adjoint.
- T_1 factors through the forgetful functor $\mathrm{SOpFib}(T1) \to \mathcal{B}/T1$.

Given a 2-monad $(\mathcal{K}, T)$ where $\mathcal{K}$ has comma objects, a strong T-morphism $f : A \to B$ is **exact** when

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \cong & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

is an exact square.

Answer to Question 1

Theorem

Let (T, η, μ) be a 2-monad on a finitely complete 2-category $\mathcal{K}$, $f : A \to B$ be a strong T-morphism between pseudo T-algebras, and V be a pseudo T-algebra. If

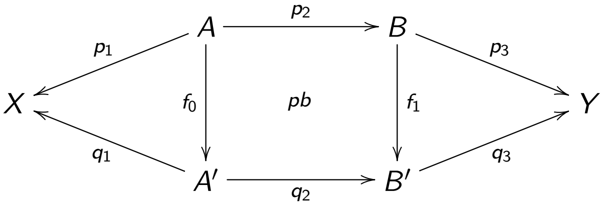
1. *T is opfamilial and μ's naturality squares are pullbacks,*
2. *f is exact, and*
3. *V is algebraically cocomplete relative to f in $\mathcal{K}$,*

then left extensions along f into V in Ps-T-Alg *are computed as in $\mathcal{K}$.*

The **polynomial functor** $\mathbf{Cat}/X \to \mathbf{Cat}/Y$ associated to a polynomial

$$p \quad : \quad X \xleftarrow{p_1} A \xrightarrow{p_2} B \xrightarrow{p_3} Y$$

in **Cat**, is the composite $\mathbf{P}(p) := \Sigma_{p_3}\Pi_{p_2}\Delta_{p_1}$. A morphism of polynomials is a diagram of the form



and induces a cartesian transformation $\mathbf{P}(p) \to \mathbf{P}(q)$.

Examples of polynomial monads. For monoids and symmetric monoidal categories:

$$1 \leftarrow \mathbb{N}_* \rightarrow \mathbb{N} \rightarrow 1 \qquad 1 \leftarrow \mathbb{P}_* \rightarrow \mathbb{P} \rightarrow 1$$

For operads:

$$\mathbb{N} \leftarrow \mathsf{RTrees}_* \rightarrow \mathsf{RTrees} \rightarrow \mathbb{N}$$

For cyclic and modular operads

$$\mathbb{N} \leftarrow \mathsf{Trees}_* \rightarrow \mathsf{Trees} \rightarrow \mathbb{N} \qquad \mathbb{N} \leftarrow \mathsf{Graphs}_* \rightarrow \mathsf{Graphs} \rightarrow \mathbb{N}$$

A **relative polynomial context**

$$\begin{array}{ccccccc}
K & \xleftarrow{r_1} & A_1 & \xrightarrow{r_2} & B_1 & \xrightarrow{r_3} & K \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \gamma_0} & \text{pb} & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle g} \\
J & \xleftarrow{q_1} & A_2 & \xrightarrow{q_2} & B_2 & \xrightarrow{q_3} & J \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi_0} & \text{pb} & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle f} \\
I & \xleftarrow[p_1]{} & A_3 & \xrightarrow[p_2]{} & B_3 & \xrightarrow[p_3]{} & I
\end{array}$$

gives rise to the following monads and monad morphisms

$$(\mathbf{Cat}/I, \mathbf{P}(p)) \underset{(\Delta_f,\phi)}{\overset{(\Sigma_f,\tilde{\phi})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathbf{Cat}/J, \mathbf{P}(q)) \underset{(\Delta_g,\gamma)}{\overset{(\Sigma_g,\tilde{\gamma})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathbf{Cat}/K, \mathbf{P}(r))$$

Answer to Question 2

Theorem

In a relative polynomial context, in which

- I, J *and* K *are discrete.*
- B_1, B_2 *and* B_3 *are groupoids.*
- p_2, q_2 *and* r_2 *are discrete fibrations with finite fibres*

the strict $\mathbf{P}(p)$*-algebra morphism*

$$\mathbf{P}(p)^{\gamma} : \mathbf{P}(p)^{\mathbf{P}(r)} \to \mathbf{P}(p)^{\mathbf{P}(q)}$$

is exact.

In fact the algebra square is a bipullback in which the vertical arrow in the generating cospan is a bi-fibration.

To prove the theorem we need a good understanding of the codescent objects involved.

$$T\tilde{F}S^2 1 \rightrightarrows\!\!\!\!\rightarrow T\tilde{F}S1 \underset{T\tilde{F}!}{\overset{\mu_{\tilde{F}1}T(\tilde{\phi}_1)}{\rightleftarrows}} T\tilde{F}1 \cdots\cdots\to T^S$$

Key features:

- The simplicial object is a category object in $\mathcal{K}$.
- $T\tilde{F}!$ is a split opfibration.
- The unit and composition maps for the category object are morphisms of split opfibrations.

enable us to compute the codescent objects more easily, and generalise the notion of crossed simplicial group.

Free operads

From

$$\begin{array}{ccccccc}
\mathbb{N} & \longleftarrow & \mathbb{N} & \longrightarrow & \mathbb{N} & \longrightarrow & \mathbb{N} \\
{\scriptstyle 1}\downarrow & & \downarrow & \text{pb} & \downarrow & & \downarrow{\scriptstyle 1} \\
\mathbb{N} & \longleftarrow & \mathsf{RTrees}_* & \longrightarrow & \mathsf{RTrees} & \longrightarrow & \mathbb{N} \\
\downarrow & & \downarrow & \text{pb} & \downarrow & & \downarrow \\
1 & \longleftarrow & \mathbb{P}_* & \longrightarrow & \mathbb{P} & \longrightarrow & 1
\end{array}$$

we express the construction of the free operad on a collection in a nice enough symmetric monoidal category V as left kan extension along $\mathrm{Sym}(\mathbb{N}) \to \mathcal{R}$, where $\mathcal{R}$ is a symmetric strict monoidal category whose objects are sequences of natural numbers, and arrows are sequences of rooted trees.

Modular envelope of a cyclic operad

From

$$\begin{array}{ccccccc}
\mathbb{N} & \longleftarrow & \mathsf{Tree}_* & \longrightarrow & \mathsf{Tree} & \longrightarrow & \mathbb{N} \\
{\scriptstyle 1}\downarrow & & \downarrow & \text{pb} & \downarrow & & \downarrow{\scriptstyle 1} \\
\mathbb{N} & \longleftarrow & \mathsf{Graph}_* & \longrightarrow & \mathsf{Graph} & \longrightarrow & \mathbb{N} \\
\downarrow & & \downarrow & \text{pb} & \downarrow & & \downarrow \\
1 & \longleftarrow & \mathbb{P}_* & \longrightarrow & \mathbb{P} & \longrightarrow & 1
\end{array}$$

we express the construction of the modular envelope of a cyclic operad in a nice enough symmetric monoidal category V as left kan extension along $\mathcal{T} \to \mathcal{G}$, where $\mathcal{T}$ and $\mathcal{G}$ are symmetric strict monoidal categories whose arrows sequences of trees and graphs respectively.

Modular envelope of a cyclic operad

Important for Kevin Costello's classification of open TCFT's as A_∞-Frobenius algebras, which comes from a homotopy equivalence between the modular envelope of a topological operad of associahedra, and an operad of moduli spaces of Riemann surfaces with boundary and marked points on the boundary.

Colimits in categories of algebras

Let C be a category. The functor

$$\mathrm{lan}_{T^\gamma} : \text{Ps-}T\text{-Alg}(T^R, V) \to \text{Ps-}T\text{-Alg}(T^S, V)$$

coming from

$$\begin{array}{ccccccc}
J \times C_0 & \xleftarrow{q_1 \times s} & A_2 \times C_1 & \xrightarrow{q_2 \times 1} & B_2 \times C_1 & \xrightarrow{q_3 \times t} & J \times C_0 \\
\downarrow g & & \downarrow \gamma_0 & \text{pb} & \downarrow \gamma_1 & & \downarrow g \\
J & \xleftarrow{q_1} & A_2 & \xrightarrow{q_2} & B_2 & \xrightarrow{q_3} & J \\
\downarrow f & & \downarrow \phi_0 & \text{pb} & \downarrow \phi_1 & & \downarrow f \\
I & \xleftarrow[p_1]{} & A_3 & \xrightarrow[p_2]{} & B_3 & \xrightarrow[p_3]{} & I
\end{array}$$

is

$$\mathrm{colim} : [C, \text{Ps-}T\text{-Alg}(T^S, V)] \to \text{Ps-}T\text{-Alg}(T^S, V).$$