FAMILIAL 2-FUNCTORS AND PARAMETRIC RIGHT ADJOINTS

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ABSTRACT. We define and study familial 2-functors primarily with a view to the development of the 2-categorical approach to operads of [Weber, 2005]. Also included in this paper is a result in which the well-known characterisation of a category as a simplicial set via the Segal condition, is generalised to a result about nice monads on cocomplete categories. Instances of this general result can be found in [Leinster, 2004], [Berger, 2002] and [Moerdijk-Weiss, 2007b]. Aspects of this general theory are then used to show that the composite 2-monads of [Weber, 2005] that describe symmetric and braided analogues of the ω-operads of [Batanin, 1998], are cartesian 2-monads and their underlying endo-2-functor is familial. Intricately linked to the notion of familial 2-functor is the theory of fibrations in a finitely complete 2-category [Street, 1974] [Street, 1980a], and those aspects of that theory that we require, that weren't discussed in [Weber, 2007], are reviewed here.

1. Introduction

The category Fam(X) of families of objects of a given category X is one of the most basic constructions in category theory. Indeed the fibration Fam(X)$\rightarrow$Set which takes a family of objects to its indexing set is one of the guiding examples for the use of fibrations to organise logic and type theory [Bénabou, 1985] [Jacobs, 1999].

We shall view the assignment $X \mapsto \text{Fam}(X)$ as the object part of an endo-2-functor of CAT. The endo-2-functor Fam has many pleasant properties. It is parametrically representable[1] in the sense of [Street, 2000]. Moreover, it preserves all types of fibrations one can define in a finitely complete 2-category [Street, 1974] [Street, 1980a]. It happens that all of Fam's pleasant properties can be derived from two axioms that can be imposed on 2-functors between finitely complete 2-categories. The general concept is that of *familial 2-functor*, and is the main subject of this paper.

The main motivation for this work is to facilitate the development of the 2-categorical approach to operads initiated in [Weber, 2005]. According to [Weber, 2005] the environment for a notion of operad is a triple $(\mathcal{K}, T, A)$ where $\mathcal{K}$ is a 2-category with products, T

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[1] Set-valued parametrically representable functors were first identified by Diers in [Diers, 1977], and called familially representable in [Carboni-Johnstone, 1995]. The general notion makes sense for functors between arbitrary categories [Weber, 2004].

is a 2-monad on $\mathcal{K}$ and A is an object of $\mathcal{K}$ which has the structure of a monoidal pseudo-T-algebra. To obtain the classical examples $\mathcal{K}$ is CAT the 2-category of categories, T is the symmetric monoidal category monad, to say that A has a monoidal pseudo-T-algebra structure is to say that A is a symmetric monoidal category, and unwinding the general definition of operad in this case gives a sequence of objects of the category A together with symmetric group actions and substitution maps as with the usual definition of operad. To obtain the operads of [Batanin, 1998] as part of this setting, one takes $\mathcal{K}$ to be the 2-category of globular categories, and T to be D_s as defined in [Batanin, 1998].

The classical theory of operads in a good symmetric monoidal category A has four basic formal aspects. First the category of symmetric sequences in A (these are sequences of objects of A with symmetric group actions) has a monoidal structure, and monoids for this monoidal structure are operads. Second is the related fact that an operad in A determines a monad on A with the same algebras as the original operad. Third is the construction of the free operad on a symmetric sequence, and fourth is the process of freely adding symmetric group actions to an operad. One may consider this last aspect as relating different operad notions (symmetric and non-symmetric). Each of these formal aspects have more complicated analogues in the theory of ω-operads [Batanin, 1998] [Batanin, 2002]. In addition to this, one has the idea of operads internal to other operads of [Batanin, 2002]. The construction of operads which are universal among those which have a certain type of internal operad structure, has been shown in [Batanin, 2002] and [Batanin, 2006] to be fundamental to the theory of loop spaces.

To give a conceptual account of these formal aspects in the general setting of [Weber, 2005] requires an understanding of how one should specialise $\mathcal{K}$, T and A. In the classical theory the formal aspects require that A have colimits that interact well with the monoidal structure. For the general setting then, it is desirable that one can discuss cocomplete objects in $\mathcal{K}$ in an efficient way. Thus $\mathcal{K}$ should be a 2-topos in the sense of [Weber, 2007]. In order for the 2-monads T in this general setting to interact well with this theory of internal colimits, they must have a certain combinatorial form. A complete discussion of this is deferred to [Weber], in which the notion of an *analytic* 2-monad on a 2-topos is defined and identified as the appropriate setting. In this paper we shall focus on those properties that an analytic 2-monad (T, η, μ) enjoys that don't involve the size issues that are encoded by the notion of 2-topos. In particular T is a familial 2-functor, $T1$ is a groupoid[2] and η and μ are cartesian. By definition such 2-monads are cartesian monads in the usual sense, but interestingly, they are also cartesian monads in a bicategorical sense (see remark(7.13) below). The general theory of operads, using the notions developed in [Weber, 2007] and this paper, will be presented in [Weber].

This paper is organised as follows. In section(2) the notions of polynomial functor and parametric right adjoint (p.r.a) are recalled. While polynomial functors are interesting and important, p.r.a functors are more general, and there are examples fundamental to higher category theory which are p.r.a but not polynomial. Then in section(3) the theory of split

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[2] An object B in a 2-category $\mathcal{K}$ is a groupoid when for all X the hom-category $\mathcal{K}(X, B)$ is a groupoid, or in other words, when every 2-cell with 0-target B is invertible.

fibrations internal to a finitely complete 2-category [Street, 1974] is recalled and in some cases reformulated in a way more convenient for us. Moreover that section includes any of the background in 2-category theory that we require, and that was not discussed already in [Weber, 2007]. Section(4) is devoted to the nerve characterisation theorem, which generalises the characterisation of categories as simplicial sets via the Segal condition to a result about a *monad with arities* on a cocomplete category. Moreover in that section applications to the theory of Dendroidal sets [Moerdijk-Weiss, 2007a] [Moerdijk-Weiss, 2007b], unpublished work of Tom Leinster [Leinster, 2004] and to the work of Clemens Berger [Berger, 2002] are provided.

A self-contained discussion of the Fam construction and the definition of familial 2-functor is presented in section(5.1). The basic properties of familial 2-functors are developed in section(6), and results enabling us to exhibit many examples of familial 2-functors are presented in section(7). Aspects of the nerve characterisation theorem are then used in section(8) to show that the unit and multiplication of the composite 2-monads of [Weber, 2005], whose corresponding operad notions include symmetric and braided versions of the higher operads of [Batanin, 1998], as cartesian.

The definitions and notations of [Weber, 2007] are used freely throughout this one. In particular, for a category $\mathbb{C}$ we denote by $\widehat{\mathbb{C}}$ the category of presheaves on $\mathbb{C}$, that is the functor category $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}]$. When working with presheaves we adopt the standard practises of writing C for the representable $\mathbb{C}(-,C) \in \widehat{\mathbb{C}}$ and of not differentiating between an element $x \in X(C)$ and the corresponding map $x : C \to X$ in $\widehat{\mathbb{C}}$. We denote by $\mathrm{CAT}(\widehat{\mathbb{C}})$ the functor 2-category $[\mathbb{C}^{\mathrm{op}}, \mathrm{CAT}]$ which consists of functors $\mathbb{C}^{\mathrm{op}} \to \mathrm{CAT}$, natural transformations between them and modifications between those. We adopt the standard notations for the various duals of a 2-category $\mathcal{K}$: $\mathcal{K}^{\mathrm{op}}$ is obtained from $\mathcal{K}$ by reversing just the 1-cells, $\mathcal{K}^{\mathrm{co}}$ is obtained by reversing just the 2-cells, and $\mathcal{K}^{\mathrm{coop}}$ is obtained by reversing both the 1-cells and the 2-cells.

When doing basic category theory, one can get a lot of mileage out of expressing one's work in terms of the paradigmatic good yoneda structure [Street-Walters, 1978] [Weber, 2007] on CAT. In particular these "yoneda methods" always seem to lead to the most efficient proofs of basic categorical facts, and we shall adopt this approach and associated notation thoughout this work. The basic construct of this structure is that given a functor $f{:}A \to B$ such that A is locally small, and for all a and b the hom $B(fa,b)$ is small, one can organise the arrow maps of f into a 2-cell

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "y_A"'] & {} \arrow[d, phantom, "\overset{\chi^f}{\Rightarrow}"] & B \arrow[dl, "B(f{,}1)"] \\
& \widehat{A} &
\end{tikzcd}$$

which is both an absolute left lifting and a pointwise left extension. See [Weber, 2007] (definition(3.1) and example(3.3)) for a fuller discussion of good yoneda structures and this example in particular. If B is itself locally small then $y_B f$ enjoys the same size condition that f did above[3], and then $\widehat{B}(y_Bf,1)$ is the functor $\widehat{B}\rightarrow\widehat{A}$ obtained by precomposing with f^{op}, and so we denote it by res_f. If in addition A is small then res_f has both adjoints. Of course this is very basic category theory, but these adjoints have useful canonical yoneda structure descriptions. The right adjoint is obtained as $\widehat{A}(B(f,1),1)$ (see [Weber, 2007] proposition(3.7)), and the left adjoint lan_f is obtained as a pointwise left extension

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\phi_f}{\Rightarrow}"] & \widehat{A} \arrow[d, "\mathrm{lan}_f"] \\
B \arrow[r, "y_B"'] & \widehat{B}
\end{tikzcd}$$

of y_Bf along y_A (see [Weber, 2007] theorem(3.20)), and since y_A is fully faithful ϕ_f is an isomorphism.

2. Polynomial functors and parametric right adjoints

In this section we recall and develop the notion of parametric right adjoint functor. Such functors have been considered by several authors at various levels of generality [Carboni-Johnstone, 1995] [Street, 2000] [Weber, 2004]. Their importance to higher dimensional algebra was first noticed by Mike Johnson [Johnson, 1987]. Parametric right adjoints are a generalisation of the more commonly studied polynomial functors [Kock] which we recall first. Recently polynomial functors have been applied in categorical study of dependent type theories [Moerdijk-Palmgren, 2000] [Gambino-Hyland, 2004] and in a conceptual description of opetopes [Batanin-Joyal-Kock-Mascari]. However, as we argue in example(2.5) below, this class of functors is not quite general enough for higher dimensional algebra, hence the need for parametric right adjoints.

Let $\mathcal{A}$ be a finitely complete category. For $f:X\rightarrow Y$ in $\mathcal{A}$ we recall that the functor $f_!:\mathcal{A}/X\rightarrow\mathcal{A}/Y$ given by composition with f, has a right adjoint denoted by f^* which is given by pulling back along f. When f^* has a further right adjoint, denoted f_*, f is said to be *exponentiable*. The category $\mathcal{A}$ is said to be *locally cartesian closed* when f_* exists for every f, and from [Freyd, 1972] we know that this is equivalent to asking that the slices of $\mathcal{A}$ are cartesian closed. Thus every elementary topos is locally cartesian closed, but CAT, the category of categories and functors is not. However in this case the exponential maps have been characterised as the Giraud-Conduché fibrations [Giraud, 1972] [Conduché, 1972], and this class of functors includes Grothendieck fibrations and opfibrations. For $X\in\mathcal{A}$ we denote by $t_X:X\rightarrow 1$ the unique map into the terminal object of $\mathcal{A}$. Obviously the functor $t_{X!}:\mathcal{A}/X\rightarrow\mathcal{A}$ has a simpler description: it takes the domain of a map into X. One may verify directly that $t_{X!}$ creates connected limits, and so for any f, $f_!$ preserves connected limits since $f_!t_{Y!}=t_{X!}$.

It is instructive to unpack these definitions in the case $\mathcal{A}=\mathrm{Set}$ because then objects of $\mathcal{A}/X$ can be regarded either as functions into X or as X-indexed families of sets via

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[3] This size condition is called *admissibility*.

the equivalence $\mathrm{Set}/X{\simeq}\mathrm{Set}^X$ which sends a function to its fibres. Thus one can describe $f_!$ as the process of taking coproducts over the fibres of f, and f_* takes cartesian products over the fibres of f.

Now from maps

$$\begin{tikzcd} W & X \arrow[l, "f"'] \arrow[r, "g"] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

in a finitely complete category $\mathcal{A}$, where g is exponentiable, one can consider the composite functor

$$\begin{tikzcd} \mathcal{A}/W \arrow[r, "f^*"] & \mathcal{A}/X \arrow[r, "g_*"] & \mathcal{A}/Y \arrow[r, "h_!"] & \mathcal{A}/Z . \end{tikzcd}$$

Functors which arise in this way are called *polynomial functors*[4]. In particular when $W = Z = 1$ we write $\mathcal{P}_g$ for the corresponding polynomial functor.

2.1. EXAMPLE. This example illustrates how polynomials with natural number coefficients can be regarded as polynomial functors. The polynomial $p(X) = X^3 + 2X + 2$ can be regarded as an endofunctor of Set: interpret X as a set, product as cartesian product of sets and sums as disjoint unions. In fact p is the following polynomial endofunctor of Set:

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "t_5^*"] & \mathrm{Set}/5 \arrow[r, "f_*"] & \mathrm{Set}/5 \arrow[r, "(t_5)_!"] & \mathrm{Set} \end{tikzcd}$$

where 5 denotes the set $\{0, 1, 2, 3, 4\}$ and $f : 5{\rightarrow}5$ is given by $f(0) = f(1) = f(2) = 0$, $f(3) = 1$ and $f(4) = 2$. To see this we trace $X \in \mathrm{Set}$ through this composite, representing objects of Set/5 as 5-tuples of sets:

$$\begin{tikzcd} X \arrow[r, mapsto, "t_5^*"] & (X, X, X, X, X) \arrow[r, mapsto, "f_*"] & (X^3, X, X, 1, 1) \arrow[r, mapsto, "(t_5)_!"] & p(X) \end{tikzcd}$$

Obtaining $f : A{\rightarrow}B$ from p is easy. Note that $p(X) = X^3+X+X+1+1$ which is a 5-fold sum and so $B = 5$, and the exponents of the terms of this sum tell us what the cardinality of the fibres of f are. In this way all polynomials with natural number coefficients can be regarded as polynomial endofunctors of the form $\mathcal{P}_f$ for f a function between finite sets, and vice versa, although different f's can give the same polynomial. For example pre- or post-composing f by a bijection doesn't change the polynomial corresponding to the induced endofunctor.

In much the same way, multivariable polynomials with natural number coefficients correspond to polynomial functors of the form $h_!g_*f^*$ where

$$\begin{tikzcd} W & X \arrow[l, "f"'] \arrow[r, "g"] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

are functions between finite sets. The cardinality of W corresponds to the number of input variables and the cardinality of Z corresponds to the number of coordinates of output. Much of the elementary mathematics of polynomials can be categorified by re-expressing in terms of polynomial functors [Kock].

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[4] In the literature authors often use the term "polynomial functor" for a special case of what we have described here. For example our polynomial functors correspond to what [Gambino-Hyland, 2004] call dependent polynomial functors, and [Kock] restricts most of the discussion to the case $\mathcal{A} = \mathrm{Set}$.

2.2. Example. For $X \in \text{Set}$ we denote by MX the free monoid on X. The elements of MX may be regarded as finite sequences of elements of X and so one has

$$MX \cong \coprod_{n \in \mathbb{N}} X^n$$

and so $M \cong \mathcal{P}_f$ where $f : \mathbb{N}_\bullet \to \mathbb{N}$,

$$\mathbb{N}_\bullet = \{(n,m) : n,m \in \mathbb{N} \text{ and } n < m\}$$

and $f(n,m) = m$. In other words the underlying endofunctor of the free monoid monad on Set is a polynomial functor. Bénabou showed that this construction works when Set is replaced by an arbitrary elementary topos with a natural numbers object [Bénabou, 1991].

For more on polynomial functors see [Kock]. We now begin our discussion of parametric right adjoints. Given a functor $T : \mathcal{A} \to \mathcal{B}$ and $X \in \mathcal{A}$ the effect of T on arrows into X may be viewed as a functor

$$T_X : \mathcal{A}/X \to \mathcal{B}/TX.$$

When $\mathcal{A}$ has a terminal object 1, we identify $\mathcal{A}/1 = \mathcal{A}$ and then the original T can be factored as

$$\mathcal{A} \xrightarrow{T_1} \mathcal{B}/T1 \xrightarrow{t_{T1!}} \mathcal{B}\,.$$

2.3. Definition. *[Street, 2000] Let $\mathcal{A}$ be a category with a terminal object. A functor $T : \mathcal{A} \to \mathcal{B}$ is a* parametric right adjoint *(henceforth p.r.a) when T_1 has a left adjoint. We shall denote such a left adjoint as L_T. A monad (T, η, μ) on $\mathcal{A}$ is p.r.a when its functor part is p.r.a and its unit and multiplication are cartesian*[5].

Notice in particular that p.r.a functors preserve pullbacks, and in fact all connected limits, since a p.r.a is a composite of a right adjoint and a functor of the form $f_!$ [Carboni-Johnstone, 1995].

2.4. Example. Polynomial functors are p.r.a: if T is of the form $h_! f_* g^*$ for f, g and h as above, then T_1 may be identified with $f_* g^*$, and so L_T is $g_! f^*$. In fact L_T is also p.r.a because $(L_T)_1$ may be identified with f^*. In particular, note that for a polynomial functor T, L_T preserves monomorphisms.

2.5. Example. We shall now give an important example of a p.r.a functor which is not polynomial. For us a graph consists of a set V of vertices, a set E of edges, and source and target functions $s, t : E \rightrightarrows V$. Thus the category Graph of graphs is the category of presheaves on the category that looks like this: $\bullet \rightrightarrows \bullet$. Let T be the endofunctor of Graph which sends $X \in \text{Graph}$ to the free category on X. That is, the vertices of TX are the same as those of X, and the edges of TX are paths in X. In particular $T1$

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[5] meaning that the naturality squares for μ and η are pullbacks.

has one vertex, and an edge for each natural number. Thus an object of Graph/$T1$ is a graph whose edges are labelled by natural numbers, and a morphism of this category is a morphism of graphs which preserves the labellings. Now applying L_T to such a labelled graph involves replacing it each edge labelled by $n \in \mathbb{N}$ by a path of length n. So when $n = 0$, one identifies the source and target of the given edge, when $n = 1$ one leaves the edge as it is, and for $n > 1$ one must add new intermediate vertices for each new path. To see that L_T really is described this way, it helps to describe it more formally as a left kan extension

$$\begin{tikzcd}
\mathbb{N} \arrow[rr, "i"] \arrow[dr, "E_T"'] & \arrow[d, phantom, "\Longrightarrow"] & \text{Graph}/T1 \arrow[dl, "L_T"] \\
& \text{Graph} &
\end{tikzcd}$$

of E_T along i, where $i(n)$ consists of a single edge labelled by n between distinct vertices, and $E_T(n) = [n]$, the graph whose vertices are natural numbers k such that $0{\leq}k{\leq}n$ and edges are $k{\to}(k{+}1)$ for $0{\leq}k{<}n$. Given this formal description it is easy to see that $L_T \dashv T_1$.

Let f be the inclusion of the vertices of the labelled graph

$$x \xrightarrow{0} y \ .$$

This is a monomorphism in Graph/$T1$, but applying L_T to f gives the unique function $\{x, y\}{\to}1$ viewed as a graph morphism between graphs with no edges. Thus $L_T(f)$ is not a monomorphism, and so by example(2.4) T is not polynomial.

It is easy to adapt example(2.5) to show that the free strict n-category endofunctor on the category of n-globular sets, and the free strict ω-category functor on the category of globular sets are p.r.a but not polynomial. The endofunctors just mentioned are fundamental to the combinatorics of higher dimensional algebra.

We shall now recall some alternative characterisations of p.r.a functors. Recall from [Weber, 2004] that $f : B{\to}TA$ is *T-generic* when for any α, β, and γ making the outside of

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[d, "f"'] & TX \arrow[d, "T\gamma"] \\
TA \arrow[r, "T\beta"'] \arrow[ur, dotted, "T\delta"] & TZ
\end{tikzcd}$$

commute, there is a unique[6] δ for which $\gamma \circ \delta = \beta$ and $T(\delta) \circ f = \alpha$. Such a δ is called a *T-filler* for the given square.

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[6] We are adopting a slight change of terminology from [Weber, 2004]. The T-generics defined here were called *strict* T-generic in [Weber, 2004], and a weaker notion in which the uniqueness of δ is dropped, are the T-generics of [Weber, 2004]. We only consider the strict notion in this paper.

2.6. Proposition. *Let $\mathcal{A}$ be a category with a terminal object and $T : \mathcal{A} \to \mathcal{B}$. Then the following statements are equivalent:*

1. *T is a parametric right adjoint.*

2. *For all $A \in \mathcal{A}$, T_A is a right adjoint.*

3. *Every $f : B \to TA$ factors as*

$$\begin{tikzcd} B \arrow[r, "g"] & TD \arrow[r, "Th"] & TA \end{tikzcd}$$

where g is T-generic.

Proof. (1)⇒(3): Let $f : B \to TA$ and then we denote $D_f = L_T(T(t_A)f)$ and $g_f : T(t_A)f \to T_1(D_f)$ as the component of the unit of $L_T \dashv T_1$ at $T(t_A)f$. Thus we have a commutative square

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g_f"'] & TA \arrow[d, "Tt_A"] \\ TD_f \arrow[r, "Tt_{D_f}"'] \arrow[ur, dotted, "Th_f"] & T1 \end{tikzcd}$$

in $\mathcal{B}$, and by the universal property of g_f as the unit, a unique h_f making the two triangles commute. It suffices to show that g_f is generic. Suppose that we have α, β and γ as in the left square

$$\begin{tikzcd} B \arrow[r, "\alpha"] \arrow[d, "g_f"'] \arrow[dr, phantom, "="] & TX \arrow[d, "T\gamma"] & & f \arrow[r, "\alpha"] \arrow[d, "g_f"'] \arrow[dr, phantom, "="] & T_1X \arrow[d, "T_1\gamma"] \\ TD_f \arrow[r, "T\beta"'] & TZ & & T_1D_f \arrow[r, "T_1\beta"'] & T_1Z \end{tikzcd}$$

which is in $\mathcal{B}$. Composing this with Tt_Z and regarding the data in $\mathcal{B}/T1$ we obtain the square on the right. The unique filler $\delta : D_f \to X$ is now obtained because of the universal property of g_f as the unit of the adjunction $L_T \dashv T_1$.
(3)⇒(2): For $f : B \to TA$ choose a factorisation

$$\begin{tikzcd} B \arrow[r, "g_f"] & TD_f \arrow[r, "Th_f"] & TA \end{tikzcd}$$

of f with g_f generic. Then the genericness of g_f implies its universal property as the unit of an adjunction with right adjoint T_A, where the object map of the left adjoint is given by $f \mapsto h_f$.
(2)⇒(1): by definition. ■

A factorisation of f as in (3) is called a *generic factorisation*. From the proof of proposition(2.6) we have:

2.7. Scholium. *A choice of L_T in proposition(2.6) amounts to a choice, for all $f : B \rightarrow TA$ of generic factorisation*

$$B \xrightarrow{g_f} TD_f \xrightarrow{Th_f} TA$$

of f, such that for all $\alpha : A \rightarrow A'$, $g_f = g_{T(\alpha)f}$ and $\alpha h_f = h_{T(\alpha)f}$. In terms of this notation, the left adjoint to T_A has object map $f \mapsto h_f$.

When working with a p.r.a functor T, we shall usually assume a choice of L_T has been made, and the notation of scholium(2.7) for the corresponding choice of generic factorisations will be used freely. In the case where T is the functor part of a p.r.a monad, these generic factorisations are in fact part of a factorisation system on $\mathrm{Kl}(T)$ [Berger, 2002], the Kleisli category of T, with every map of $\mathrm{Kl}(T)$ factoring as a generic map followed by a *free* map. The free maps are those which are in the image of the identity on objects left adjoint functor $\mathcal{A} \rightarrow \mathrm{Kl}(T)$. We shall reflect these circumstances in the notation we use: writing the chosen generic factorisation of a map $f : B \rightarrow A$ in $\mathrm{Kl}(T)$ as

$$B \xrightarrow{g_f} D_f \xrightarrow{h_f} A$$

where g_f is generic and h_f is free. Moreover we shall assume that our choice of generic factorisations is *normalised*: namely that when f is itself a free map $D_f = B$, $g_f = \mathrm{id}$ and $h_f = f$. This is clearly always possible and is a useful simplification. Indeed, another way to express the condition that our choice of generic factorisations is normalised, is to say that all the components of the unit η are chosen generics.

2.8. Example. Let T be the free category endofunctor of Graph and recall the graphs $[n]$ from example(2.5). To give a graph morphism $f : [1] \rightarrow TX$ is to give a path in X, that is, a graph morphism $p : [n] \rightarrow X$. Writing $g : [1] \rightarrow T[n]$ for the path in $[n]$ starting at 0 and finishing at n (ie the maximal path), we have a factorisation

$$[1] \xrightarrow{g} T[n] \xrightarrow{Tp} TX$$

of f which the reader may easily verify is the generic factorisation of f.

2.9. Example. Let $p : E \rightarrow B$ be a discrete fibration between small categories. Also denote by p the presheaf on B corresponding to $p : E \rightarrow B$ via the Grothendieck construction. Then the functor $\mathrm{lan}_p : \widehat{E} \rightarrow \widehat{B}$ obtained by left kan extension along p^{op}, factors as the canonical equivalence $\widehat{E} \simeq \widehat{B}/p$ followed by the domain functor $\widehat{B}/p \rightarrow \widehat{B}$. We may identify this canonical equivalence with $(\mathrm{lan}_p)_1$ and so lan_p is p.r.a.

Parametric right adjoints between presheaf categories are particularly easy to understand. In particular, to verify p.r.a'ness of a functor T one need only generically factor maps with whose domain is representable and codomain is $T1$.

2.10. Proposition. *For a functor $T : \widehat{\mathbb{B}} \to \widehat{\mathbb{C}}$ the following statements are equivalent:*

1. *Any $f : C \to T1$ has a generic factorisation where $C \in \mathbb{C}$ (is regarded as a representable presheaf).*
2. *T is p.r.a.*

Proof. (2)⇒(1) by proposition(2.6). On the other hand given (1) one can define the functor

$$E_T : y/T1 \to \widehat{\mathbb{B}} \qquad x : C \to T1 \;\mapsto\; D_x$$

and by the definition of "generic" the assignment

$$z : C \to TZ \;\mapsto\; h_z : D_z \to X$$

and scholium(2.7) gives bijections

$$T(Z)(C) \cong \coprod_{x \in T1(C)} \widehat{\mathbb{C}}(E_T(x), Z). \tag{1}$$

Thus $T_1 \cong \widehat{\mathbb{C}}(E_T, 1)$ and so L_T is obtained as the left kan extension of E_T along the composite of the yoneda embedding and the canonical equivalence $\widehat{y/T1} \simeq \widehat{\mathbb{C}}/T1$. ■

2.11. Remark. From the definition of $E_T : y/T1 \to \widehat{\mathbb{B}}$ in the above proof and lemma(5.7) of [Weber, 2004], an object $X \in \widehat{\mathbb{B}}$ is isomorphic to an object in the image of E_T iff there exists $C \in \mathbb{C}$ a generic morphism $C \to TX$.

2.12. Remark. In the proof of proposition(2.10) we reduced the data for a p.r.a $T : \widehat{B} \to \widehat{C}$ to the functor $E_T : y/T1 \to \widehat{B}$. Denoting by $p : E \to C$ the discrete fibration corresponding to $T1$ by the Grothendieck construction (so $E = y/T1$), T amounts to a span

$$\begin{tikzcd} \widehat{B} & E \arrow[l, "e"'] \arrow[r, "p"] & C \end{tikzcd}$$

(ie $e = E_T$), and one can recapture T from such a span as the composite $\mathrm{lan}_p \widehat{B}(e, 1)$: lan_p is p.r.a by example(2.9) and $\widehat{B}(e, 1)$ has left adjoint given by left kan extension as in the proof of proposition(2.10). Now a functor $e : E \to \widehat{B}$ can in turn be regarded as a 2-sided discrete fibration with small fibres [Street, 1974] [Street, 1980a] [Weber, 2007], and writing

$$\begin{tikzcd} B & D \arrow[l, "d"'] \arrow[r, "c"] & E \end{tikzcd}$$

for the underlying span in Cat of this discrete fibration we see that the data for a p.r.a functor between presheaf categories consists of functors

$$\begin{tikzcd} B & D \arrow[l, "d"'] \arrow[r, "c"] & E \arrow[r, "p"] & C \end{tikzcd}$$

between small categories such that the pair (d, c) form a discrete fibration from B to E, and p is a (one-sided) discrete fibration. In the case where the categories in question are all discrete, T is just the polynomial functor $p_!c_*d^*$. Hence if $T : \widehat{B} \to \widehat{C}$ and B and C are discrete, then T is p.r.a iff T is polynomial. Thus we have exhibited p.r.a functors between presheaf categories as a natural generalisation of polynomial functors defined from Set.

Moreover p.r.a functors between presheaf categories have been characterised

2.13. Theorem. *[Weber, 2004] Let B and C be small categories. A functor $T : \widehat{B} \to \widehat{C}$ with rank is p.r.a iff it preserves connected limits.*

although we shall not make much use of this characterisation in this paper. Instead we use the explicit descriptions provided by proposition(2.10) and remark(2.12). When giving a self-contained discussion of any particular example of a p.r.a monad T on a presheaf category $\widehat{\mathbb{C}}$, guided by these results one orders the discussion in the following way. First one describes $T1$ and constructs the functor E_T. Then T is obtained as in the above remark. With these descriptions in hand one then is in a position to describe explicitly the unit and multiplication of the monad. Implicitly we followed this approach already in example(2.5) for the category monad on Graph, and in [Batanin, 1998] this was the way in which the strict ω-category monad on globular sets was described. To illustrate further we will now discuss in detail an example which is central to [Moerdijk-Weiss, 2007a] and [Moerdijk-Weiss, 2007b]. In particular one should compare the following example to the discussion in section(3) of [Moerdijk-Weiss, 2007a].

2.14. Example. A *multigraph* consists of a set of vertices and a set of multi-edges. A multi-edge goes from an n-tuple of vertices to a single vertex, and a typical such is depicted as

$$f : (a_1, ..., a_n) \to b$$

where a_i and b are vertices. When $n = 1$ f is called an edge. A *coloured operad*[7] is a multigraph X admitting:

1. units: for each vertex a, one has an edge $1_a : (a) \to a$.
2. symmetric group actions: for all $n \in \mathbb{N}$ and vertices a_i and b for $1 \leq i \leq n$, one has an action of the n-th symmetric group on the set $X(a_1, ..., a_n; b)$ of multi-edges from $(a_1, ..., a_n)$ to b.
3. substitution of multi-edges: as in a multicategory satisfying associativity and unit laws, and equivariant with respect to the symmetric group actions.

Informally the free coloured operad TX on a multigraph X can be described as follows. The vertices are the same as those of X, and a multi-edge of TX may be pictured as a

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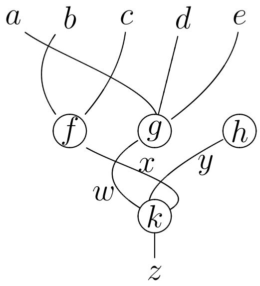
[7] Coloured operads are also called symmetric multicategories, or as in [Moerdijk-Weiss, 2007a] [Moerdijk-Weiss, 2007b], are sometimes just referred to as operads.

tree-with-permutations labelled in a compatible way by the vertices and multi-edges of X. To illustrate, here is a picture of a multi-edge $(a,b,c,d,e) \to z$ in TX:



where

$$f : (b,c) \to x \qquad g : (a,d,e) \to w \qquad h : () \to y \qquad k : (w,y,x) \to z$$

are multi-edges in X. We will now formalise this, and along the way exhibit T as a p.r.a monad on the presheaf category of multigraphs.

Denote by $\mathbb{M}$ the following category. There is an object 0 and for $n \in \mathbb{N}$ an object $(n,1)$. For each $n \in \mathbb{N}$ there is an arrow $\tau_n : 0 \to (n,1)$, and for $1 \leq i \leq n$ there are arrows $\sigma_{n,i} : 0 \to (n,1)$. There are no relations. Clearly an object X of $\widehat{\mathbb{M}}$ is a multigraph: the elements of $X(0)$ are the vertices of X and the elements $X(n,1)$ are the multi-edges with source of length n.

The multigraph $T1$ has one object which E_T sends to the representable multi-graph 0. Informally the multi-edges of T1 are trees-with-permutations. We now give an inductive definition of the multi-edges of $T1$ and their images by the functor E_T. For each $p \in T1(n,1)$ we must also define objects $(\sigma_1, ..., \sigma_n, \tau)$ of $E_T(p)$ in order that the arrow map of the functor E_T be defined. We refer to σ_i as the *i-th source* of p and τ as the *target* of p. The initial step of the definition is that $T1(1,1)$ has an element which gets sent to the representable $0 \in \widehat{\mathbb{M}}$ by E_T, and so σ_1 and τ for this element are both equal to the unique object of 0. By induction $p \in T1(n,1)$ is a triple (q,f,ρ), where $q \in T1(k)$,

$$f : \{1, ..., n\} \to \{1, ..., k\}$$

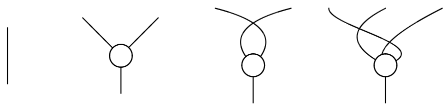
is an order preserving function, and ρ is a permutation of n symbols. The multi-graph $E_T(p)$ contains $E_T(q)$ and has the same target, and in addition there are n new objects $(a_1, ..., a_n)$, and k new multi-edges

$$(a_j : \alpha(i) \leq j \leq \beta(i)) \to \sigma'_i$$

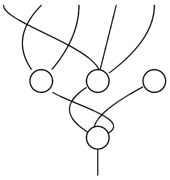
where σ'_i is the i-th source of q and $\alpha(i)$ (resp. $\beta(i)$) is the least (resp. greatest) element of the fibre $f^{-1}(i)$. The i-th source of p is given by $\sigma_i = a_{\rho i}$. Having described $T1$ and E_T one now obtains a description of TX for any $X \in \widehat{\mathbb{M}}$ from equation(1).

Here are some examples to enable the reader to reconcile the formal description of $T1$ just given with the intuitive idea of trees-with-permutations. The representables 0 and

$(2,1) = (0, t_2, \mathrm{id})$, $(0, t_2, (12))$, and $q = (0, t_3, (132))$ are pictured as



respectively, and $(q, f, (132))$ where $f : \{1,2,3,4,5\} \to \{1,2,3\}$ is given by $f(1) = f(2) = 1$ and $f(3) = f(4) = f(5) = 2$, is drawn as

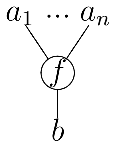


and is an element of $T1(5,1)$. The multigraph $E_T p$ has one object for each edge of the tree and a multi-edge of arity k for each node with k edges coming in from above. So in this last example $E_T p$ has 9 objects, a multi-edge of arity 0, a multi-edge of arity 2, and two multi-edges of arity 3. Thus equation(1) for TX expresses formally the intuition of trees-with-permutations labelled by X: by the equation a multi-edge of TX consists of $p \in T1(n,1)$ together with a morphism $E_T p \to X$ in $\widehat{\mathbb{M}}$, and such a morphism amounts to a labelling of the edges of the tree p by the objects of X. By definition T is a p.r.a endofunctor of $\widehat{\mathbb{M}}$.

One reconciles the above discussion of trees with that of [Moerdijk-Weiss, 2007a], in which non-planar trees are used, by observing that in an obvious way one can associate a non-planar tree to any multi-edge of $T1$, and p and q have the same associated non-planar tree iff $E_T p \cong E_T q$. Thus one has a bijection between non-planar trees and isomorphism classes of multigraphs in the image of E_T.

There is an inclusion $\eta_X : X \to TX$ of multigraphs which is the identity on objects, and on arrows is given by

$f : (a_1, ..., a_n) \to b \quad \mapsto$



Grafting of labelled trees provides substitution for TX, and permuting input edges of labelled trees provides symmetric group actions for the $TX(n,1)$ with respect to which multi-composition is equivariant. That is, TX is a coloured operad in an obvious way. Moreover the reader will easily verify[8] directly that composition with η_X gives a bijection between morphisms $TX \to Z$ of coloured operads and morphisms $X \to Z$ of multigraphs, and so TX really is the free coloured operad on X. The monad multiplication of T encodes substitution of trees: given a tree whose nodes are labelled by labelled trees, that

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[8] Indeed the definition of TX described here was selected so that this universal property follows pretty much by definition.

is a multi-edge of $T^2(X)$, applying μ_X amounts to substituting the labels and removing the outer nodes. One may verify directly that μ_X and η_X are natural in X, and that for each X one has

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "t_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "Tt_X"] \\ 1 \arrow[r, "\eta_1"'] & T1 \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "T^2t_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "Tt_X"] \\ T^21 \arrow[r, "\mu_1"'] & T1 \end{tikzcd}$$

and so (T, η, μ) is indeed a p.r.a monad on $\widehat{\mathbb{M}}$.

The remainder of this section is devoted to two lemmas, which are useful technical consequences of parametric right adjointness. In fact the second of these results only requires that the functor in question preserves pullbacks. Let $T : \mathcal{A} \to \mathcal{B}$ be a functor between categories which have products, and

$$(A_i : i \in I)$$

be a family of objects of $\mathcal{A}$. Writing p_{A_i} for the i-th projection of the product, one has the comparison map

$$T(\prod A_i) \xrightarrow{k_{T,A_i}} \prod T(A_i)$$

defined by $p_{T(A_i)} k_{T,A_i} = T(p_{A_i})$. In general this comparison map is natural in the A_i, however when T is p.r.a slightly more than this is true.

2.15. Lemma. *Let $T : \mathcal{A} \to \mathcal{B}$ be a p.r.a functor between categories which have products. Then the comparison maps k_{T,A_i} just defined are* cartesian *natural*[9] *in the A_i.*

Proof. Given functors

$$\mathcal{A} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{C}$$

between categories with products, if both S and T have cartesian natural product comparison maps and S preserves pullbacks, then the composite ST will have cartesian natural product comparison maps because these maps are given by the composites

$$ST(\prod A_i) \xrightarrow{Sk_T} S\prod T(A_i) \xrightarrow{k_S} ST\prod A_i$$

for each family $(A_i : i \in I)$ of objects of $\mathcal{A}$. Thus since any p.r.a functor is a composite of a right adjoint and a functor of the form $t_{X,!} : \mathcal{A}/X \to \mathcal{A}$, it suffices to show that the lemma holds for this last special case, and this is straight-forward to verify directly. ∎

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[9] meaning that the commuting squares that witness the naturality in the given variables are in fact pullbacks.

The final result of this section is the analogue of lemma(2.15) for *co*products. Recall that a category $\mathcal{E}$ is *small-extensive* [Carboni-Lack-Walters, 1993] when it has coproducts and when for each family of objects $(X_i : i \in I)$ of $\mathcal{E}$ the functor

$$\prod_{i \in I} \mathcal{E}/X_i \to \mathcal{E}/\left(\sum_{i \in I} X_i\right)$$

which sends a family of maps $(h_i : Z_i {\to} X_i)$ to their coproduct is an equivalence. There are many examples of small-extensive categories: for instance every Grothendieck topos is small extensive, as is CAT. For a family of objects of $\mathcal{E}$ as above we shall denote the i-th coproduct inclusion as c_{X_i}. Let $(f_i : X_i {\to} Y_i)$ be a family of maps and recall that small-extensivity implies that each of the squares

$$\begin{tikzcd}
X_i \arrow[r, "c_{X_i}"] \arrow[d, "f_i"'] & \coprod X_i \arrow[d, "\coprod f_i"] \\
Y_i \arrow[r, "c_i"'] & \coprod Y_i
\end{tikzcd}$$

are pullbacks, and that given an arrow $g : A {\to} B$ and a family of pullback squares as shown on the left,

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d, "f_i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "g"] & & \coprod X_i \arrow[r] \arrow[d, "\coprod f_i"'] & A \arrow[d, "g"] \\
Y_i \arrow[r] & B & & \coprod Y_i \arrow[r] & B
\end{tikzcd}$$

then the induced square shown on the right in the previous display is also a pullback. Let $T : \mathcal{A} {\to} \mathcal{B}$ be a functor between small-extensive categories, and $(A_i : i \in I)$ be a family of objects of $\mathcal{A}$. One has the comparison map

$$\coprod TA_i \xrightarrow{k_{T,A_i}} T(\coprod A_i)$$

defined by $k_{T,A_i} c_{T(A_i)} = T(c_{A_i})$. As an immediate consequence of the definitions and the basic consequences of small extensivity that we have just recalled, one has

2.16. Lemma. *Let $T : \mathcal{A} {\to} \mathcal{B}$ be a pullback preserving functor between small-extensive categories. Then the comparison maps k_{T,A_i} just defined are* cartesian *natural in the A_i.*

3. Some 2-categorical background

In this section we recall the descriptions of split fibrations, iso-fibrations and bicategorical fibrations internal to a finitely complete 2-category $\mathcal{K}$. This theory is due to Ross Street [Street, 1974] [Street, 1980b]. We shall also provide a mild reformulation of Street's theory of split fibrations that is useful for this work.

3.1. Split fibrations. One has a notion of (Grothendieck) fibration in any finitely complete 2-category [Street, 1974]. The definition is based on the idea of a *cartesian* 2-cell. For a functor $f : A \to B$, recall that a morphism $\alpha : a_1 \to a_2$ in A is f*-cartesian* when for all α_1 and β as shown:

$$\begin{tikzcd}
fa_1 \arrow[r, "f\alpha"] & fa_2 \\
fa_3 \arrow[u, "\beta"] \arrow[ur, "f\alpha_1"'] \arrow[ur, phantom, "=" near end, shift left=3] &
\end{tikzcd}$$

there is a unique $\gamma : a_3 \to a_1$ such that $f\gamma = \beta$ and $\alpha\gamma = \alpha_1$. Then for $f : A \to B$ in $\mathcal{K}$ one says that a 2-cell

$$\begin{tikzcd}
X \arrow[r, bend left, "a_1"{name=U}] \arrow[r, bend right, "a_2"'{name=D}] & A
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

is f*-cartesian* when for all $g : Y \to X$ in $\mathcal{K}$, $\alpha g \in \mathcal{K}(Y, A)$ is $\mathcal{K}(Y, f)$-cartesian. A fibration in $\mathcal{K}$ is then defined to be a map $f : A \to B$ such that for all

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "b"'] & {} \arrow[d, phantom, "\overset{\beta}{\Rightarrow}" near start] & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

there exists f-cartesian $\overline{\beta} : c \Rightarrow a$ so that $fc = b$ and $f\overline{\beta} = \beta$. That is every β as shown here has a "cartesian lift" $\overline{\beta}$.

There is a nice characterisation of fibrations: f is a fibration iff the canonical map $\eta_f : A \to B/f$ has a right adjoint over B. In general, this adjunction $\eta_f \dashv a$ should be regarded as a *choice* of cartesian liftings, just as in the familiar case $\mathcal{K} = \mathrm{CAT}$, and is referred as a *cleavage* for f. It is convenient to denote cleavages as ordered pairs (a, ε), where ε is counit of $\eta_f \dashv a$.

To make the choice of liftings from a given cleavage explicit write

$$\begin{tikzcd}
B/f \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

for the defining lax pullback square of B/f and let (a, ε) be a cleavage for f. Given a 2-cell β as above, one defines its cartesian lift $\overline{\beta}$ to be $q\varepsilon\beta'$ where β' is unique such that

$$\beta \quad = \quad \begin{tikzcd}
X \arrow[dr, "\beta'"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] \arrow[dr, phantom, "=" , shift left=4] & & \\
& B/f \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] \arrow[l, phantom, "=" near start] & A \arrow[d, "f"] \\
& B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

One may show that the 2-cell $q\varepsilon$ is f-cartesian and so $q\varepsilon\beta'$ is f-cartesian for any β. This choice of cartesian liftings has the following property: for any $x : Z{\rightarrow}X$, $\overline{\phi x} = \overline{\phi}x$. Conversely given such a choice of cartesian liftings, one obtains a cleavage by lifting the defining pullback square for B/f and using its universal property to induce the counit $\varepsilon : \eta_f a{\rightarrow}1$ and so one has:

3.2. Lemma. *To give a cleavage for $f : A{\rightarrow}B$ is the same as giving for all 2-cells*

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "b"'] & \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

a choice $\overline{\phi}$ of f-cartesian lift of ϕ such that $\overline{\phi x} = \overline{\phi}x$ for all 1-cells $x : Z{\rightarrow}X$.

Thus amongst those 2-cells whose 0-target is A, there are those f-cartesian 2-cells which are chosen by the given cleavage for f. In the special case $\mathcal{K} = \mathrm{CAT}$, it is enough to consider $X = 1$ and then one speaks of *chosen-cartesian* arrows of A.

3.3. Definition. *Let $f : A{\rightarrow}B$ be a fibration in $\mathcal{K}$ and be a cleavage (a, ε) for f. A 2-cell*

$$\begin{tikzcd}
X \arrow[r, bend left, "a_1"{name=U}] \arrow[r, bend right, "a_2"'{name=D}] & A \arrow[phantom, from=U, to=D, "\Downarrow\phi"]
\end{tikzcd}$$

is chosen-(a, ε)-cartesian *when the cartesian lift of $f\phi$ provided by the cleavage (a, ε) is ϕ. When a cleavage (a, ε) for f is given, we will often abuse terminology and say that ϕ is* chosen-f-cartesian *when there is little risk of confusion.*

Thus one can define split fibrations in $\mathcal{K}$ in an elementary way by analogy with $\mathcal{K} = \mathrm{CAT}$. Recall that in this special case a cleavage is split when: (1) identity arrows are chosen-cartesian, and (2) chosen-cartesian arrows are closed under composition.

3.4. Definition. *A cleavage (a, ε) for $f : A{\rightarrow}B$ is* split *when for the resulting chosen-cartesian 2-cells we have:*

1. *For any $k : X{\rightarrow}A$, the identity 2-cell 1_k is chosen-f-cartesian.*
2. *A vertical composite of chosen-f-cartesian 2-cells is chosen-f-cartesian.*

A split fibration *in $\mathcal{K}$ is a fibration $f : A{\rightarrow}B$ which has a split cleavage for f.*

One can define the 2-category $\mathrm{Spl}(B)$ of split fibrations over B as follows:

- objects consist of a map $f : A{\rightarrow}B$ and a split cleavage (a, ε) for f.
- an arrow of $\mathrm{Spl}(B)$ is an arrow in $\mathcal{K}/B$ which preserves chosen-cartesian 2-cells.
- a 2-cell of $\mathrm{Spl}(B)$ is just a 2-cell in $\mathcal{K}/B$.

and denote the forgetful 2-functor by $U_B : \mathrm{Spl}(B) \to \mathcal{K}/B$. In Street [Street, 1974] split fibrations over B are defined a little differently to definition(3.4): as the strict algebras for the 2-monad on $\mathcal{K}/B$ whose underlying endofunctor

$$\Phi_B : \mathcal{K}/B \to \mathcal{K}/B$$

obtained by lax pullback along 1_B. The reader will easily supply the formal definitions of the unit and multiplication of this monad, and verify that it is in fact a cartesian monad. More importantly from [Street, 1974], we know that Φ_B is colax idempotent (or in older terminology, Kock-Zöberlein of limit-like variance). We omit the straight-forward proof of the following result, which reconciles the two definitions of split fibrations we have been discussing.

3.5. THEOREM. *Let $\mathcal{K}$ be a finitely complete 2-category and $B \in \mathcal{K}$. There is an isomorphism* $\mathrm{Spl}(B) \cong \Phi_B\text{-}\mathrm{Alg}_s$ *commuting with the forgetful functors into $\mathcal{K}/B$.* [10]

The basic, elementary, well-known and easy to prove facts about split fibrations at this level of generality are summarised in

3.6. THEOREM.

1. *A composite of split fibrations is a split fibration.*
2. *The pullback of a split fibration along any map is a split fibration.*
3. *If $A \xleftarrow{d} E \xrightarrow{c} B$ is a discrete fibration from A to B then d is a split fibration.*

and in light of the fact that split fibrations over a given base form a 2-category, with little more effort one can verify the following more comprehensive expression the pullback stability.

3.7. THEOREM. *Let $\mathcal{K}$ be a finitely complete 2-category and $f : A \to B$ be a morphism of f. The 2-functor f^* given by pulling back along f lifts as in the square on the left,*

$$\begin{tikzcd}
\mathrm{Spl}(B) \arrow[r, dotted] \arrow[d, "U_B"'] \arrow[dr, phantom, "="] & \mathrm{Spl}(A) \arrow[d, "U_A"] \\
\mathcal{K}/B \arrow[r, "f^*"'] & \mathcal{K}/A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spl}(A) \arrow[r, dotted] \arrow[d, "U_A"'] \arrow[dr, phantom, "="] & \mathrm{Spl}(B) \arrow[d, "U_B"] \\
\mathcal{K}/A \arrow[r, "f_!"'] & \mathcal{K}/B
\end{tikzcd}$$

and when f is a split fibration, the 2-functor $f_!$ given by composition with f lifts as in the square on the right.

Of course for a locally discrete 2-category $\mathrm{Spl}(B)$ and $\mathcal{K}/B$ coincide and everything discussed in this subsection becomes trivial. For instance Φ_B is the identity monad. Thus one may regard $\mathrm{Spl}(B)$ as the "real" 2-categorical slice over B, with $\mathcal{K}/B$ its "naive" cousin. This point of view will become important in section(5).

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[10] Recall that for a 2-monad T, $T\text{-}\mathrm{Alg}_s$ denotes the 2-category of strict algebras, strict algebra morphisms and algebra 2-cells (see [Blackwell-Kelly-Power, 1989]).

3.8. Iso-fibrations. An *iso-fibration* in a finitely complete 2-category $\mathcal{K}$ is a map $f : A \to B$ such that for all invertible 2-cells

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "b"'] & \overset{\beta}{\Rightarrow} & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

there exists an invertible $\overline{\beta} : c \Rightarrow a$ so that $fc = b$ and $f\overline{\beta} = \beta$. For example, any fibration and any opfibration is an iso-fibration since the (op)cartesian lift of an invertible 2-cell is invertible. Another important example is provided by

3.9. Example. Let $\mathcal{K}$ be a finitely complete 2-category, $A \in \mathcal{K}$ and (t, η, μ) a monad on A in $\mathcal{K}$. Then the underlying forgetful arrow $u : A^t \to A$ of the Eilenberg-Moore object of t is a *discrete* iso-fibration: given $b : X \to A^t$, $a : X \to A$ and $\beta : a \cong ub$ there exists a *unique* $\overline{\beta} : c \cong a$ so that $u\overline{\beta} = \beta$. To see this note that by the representability of the notions involved it suffices to consider the case $\mathcal{K} = \mathrm{CAT}$ in which case u is just the forgetful functor from the category of t-algebras. In this case the assertion amounts to the fact that for any isomorphism between an object of $a \in A$ and the underlying object of a t-algebra, one has a unique t-algebra structure on a with respect to which the given isomorphism lives in t-Alg.

One has the following analogue of the adjoint characterisation of a fibration, which one can prove by adapting the proof of its analogue by assuming that all 2-cells of that proof are invertible.

3.10. Proposition. *Let $\mathcal{K}$ be a finitely complete 2-category and $f : A \to B$ in $\mathcal{K}$. Then the following statements are equivalent:*

1. *f is an iso-fibration.*
2. *For all $g : X \to B$, the map $i : g/_{=}f \to g/_{\cong}f$ has a pseudo inverse in $\mathcal{K}/X$.*
3. *The map $i : A \to B/_{\cong}f$ has a pseudo inverse in $\mathcal{K}/B$.*

Moreover when B is a groupoid the following are equivalent for $f : A \to B$:

- f is a fibration.
- f is an opfibration.
- f is an iso-fibration.

because a 2-cell ϕ with 0-target A is f-cartesian iff ϕ is invertible iff ϕ is f-opcartesian. Thus a cleavage for the fibration f, thought of as a choice of f-cartesian 2-cells as in lemma(3.2), is also a cleavage for f seen as an opfibration. Thus we have an isomorphism $\mathrm{Spl}(B) \cong \mathrm{SplOp}(B)$ commuting with the forgetful 2-functors into $\mathcal{K}/B$.

3.11. Bipullbacks. Taking the representable definition of pseudo pullback one can ask instead that the induced functor be an equivalence. In a finitely complete 2-category this is equivalent to the following definition. A *bipullback* of

$$\begin{tikzcd} A \arrow[r, "f"] & B & C \arrow[l, "g"'] \end{tikzcd}$$

in a finitely complete 2-category $\mathcal{K}$ consists of an invertible 2-cell

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\cong"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in $\mathcal{K}$ such that the arrow $P \rightarrow f/_{\cong}g$ it induces is an equivalence.

3.12. Example. While it is not true in general that a pullback square

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in $\mathcal{K}$ is a bipullback, it is true whenever g is an isofibration by proposition(3.10).

It is well-known and straight-forward to prove that bipullbacks satisfy an analogous composition/cancellation result to those enjoyed by lax and pseudo pullbacks.

3.13. Proposition. *Let $\mathcal{K}$ be a finitely complete 2-category and*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & A \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & C \arrow[d, "g"] \\ Y \arrow[r, "h"'] & B \arrow[r, "f"'] & D \end{tikzcd}$$

be invertible 2-cells in $\mathcal{K}$ such that ψ exhibits A as a bipullback of f and g. Then ϕ exhibits X as a bipullback of u and h iff the composite isomorphism exhibits X as a bipullback of fh and g.

3.14. Bifibrations. While isomorphisms of categories are obviously fibrations, equivalences of categories may not be. For a simple example recall the functor ch : SET→CAT which sends any set X to the category whose objects are the elements of X, and for which there is a unique morphism between any two objects of ch(X). Clearly if X is non-empty then ch(X)$\simeq$1, and ch sends functions between non-empty sets to equivalences of categories. Apply ch to a function from a one element set to a two element set for an example of an equivalence which is not a fibration[11]. Thus the concept of fibration described so far is 2-categorical in nature. However there is an analogous bicategorical concept [Street, 1980b], which we refer to as *bifibration*[12], that we shall now discuss.

We content ourselves with simply providing the elementary definition of bifibration in a finitely complete 2-category, proving that bifibrations compose and are stable under bipullback, and providing an adjoint characterisation. In [Street, 1980b] much more than this is achieved: in particular a pseudo-monadic description of bifibrations is obtained analogous to the description of fibrations as pseudo algebras.

Let $\mathcal{K}$ be a finitely complete 2-category. A map $f : A \to B$ in $\mathcal{K}$ is a *bifibration* when for all

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "b"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

there exists f-cartesian $\overline{\beta} : c \Rightarrow b$ and invertible $\iota : b \Rightarrow fc$ such that

$$\beta \quad = \quad \begin{tikzcd} X \arrow[rr, bend left, "a"] \arrow[rr, bend right, "c"'] \arrow[rr, phantom, "\overset{\overline{\beta}}{\Uparrow}"] \arrow[dr, "b"'] & \arrow[d, phantom, "\overset{\iota}{\Rightarrow}"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

3.15. EXAMPLES.

1. By definition fibrations are bifibrations.

2. Equivalences are bifibrations. Let $f : A \to B$ be an equivalence. Thus there is an adjunction $i \dashv f$ whose unit η and counit ε are invertible. Moreover f is representably fully faithful. One may check easily that for any fully faithful functor $g : X \to Y$, all the morphisms of X are g-cartesian. Thus by definition all 2-cells between arrows with codomain A are f-cartesian. Now observe that the equations

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "b"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[r, "1_A"] \arrow[d, "f"] \arrow[dr, phantom, "\overset{\varepsilon}{\Uparrow} \quad \overset{\eta}{\Uparrow}"] & A \arrow[d, "f"] \\ & B \arrow[r, "1_B"'] \arrow[ur, "i"] & B \end{tikzcd}$$

for all β, exhibit f as a bifibration since η is invertible.

3. An object B of $\mathcal{K}$ is said to be a *groupoid* when for all $X \in \mathcal{K}$, the hom-category $\mathcal{K}(X, B)$ is a groupoid in CAT. Any map $f : A \to B$ where B is a groupoid is a bifibration, because any 2-cell β as in the above definition of bifibration is invertible.

The basic facts on bifibrations are

---

[11] In fact this functor is not even a Giraud-Conduché fibration, so we have here an example of an equivalence of categories which is not even an exponentiable functor.

[12] Although in the literature the word "bifibration" is sometimes taken to mean a functor which is both a fibration and a opfibration, we prefer to stick to the convention of using the prefix "bi" for bicategorical notions.

3.16. Proposition. *[Street, 1980b]*

1. *The composite of bifibrations in a finitely complete* 2*-category is a bifibration.*

2. *Bifibrations are stable under bipullback.*

Finally one can also give an adjoint characterisation of bifibrations by adapting the proof of the usual adjoint characterisation of fibrations in the obvious way. In this case this criterion establishes a nice relationship between forming lax pullbacks and forming *pseudo* pullbacks along a bifibration. Thus for $f : A \to B$ and $g : X \to B$, we denote by $i : g/_{\cong}f \to g/f$ the arrow induced by the defining isomorphism

$$\begin{tikzcd} g/_{\cong}f \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[d, "f"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

for $g/_{\cong}f$, and the universal property of g/f.

3.17. Theorem. *[Street, 1980b] Let $\mathcal{K}$ be a finitely complete* 2*-category and $f : A \to B$ in $\mathcal{K}$. Then the following statements are equivalent:*

1. *f is a bifibration.*

2. *For all $g : X \to B$, the map $i : g/_{\cong}f \to g/f$ has a right adjoint in $\mathcal{K}/X$ with invertible unit.*

3. *The map $i : B/_{\cong}f \to B/f$ has a right adjoint in $\mathcal{K}/B$ with invertible unit.*

4. Abstract nerves

In this section we analyse the monad theory which provides a far-reaching generalisation of the characterisation of categories as simplicial sets satisfying the Segal condition. A general theorem in this vein is an unpublished result of Tom Leinster [Leinster, 2004], and applies to a p.r.a monad T on a presheaf category. Here we describe a more general result and I am grateful to Steve Lack for suggesting the general setting expressed in definition(4.1). Throughout this section we make serious use of the yoneda structures notation discussed in the introduction.

4.1. Definition. *A* monad with arities *consists of a monad (T, η, μ) on a cocomplete category A together with a fully faithful and dense functor*

$$i_0 : \Theta_0 \to A$$

such that Θ_0 *is small, and the functor* $A(i_0, T)$ *preserves the left extension*

$$\begin{tikzcd}
\Theta_0 \arrow[rr, "i_0"] \arrow[dr, "i_0"'] & {} \arrow[d, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & A \arrow[dl, "1"] \\
& A &
\end{tikzcd}$$

We shall denote such a monad with arities as a pair (T, Θ_0).

4.2. Remark. Since $A(i_0, 1)$ is fully faithful and fully faithful maps reflect left extensions, T will automatically preserve the above left extension since $A(i_0, T)$ does.

4.3. Examples. One can exhibit any accessible monad T on a locally presentable category A as a monad with arities. In this situation there is a regular cardinal α such that A is locally α-presentable and T preserves α-filtered colimits, and then one takes i_0 to be the inclusion of the α-presentable objects. The functor $A(i_0, T)$ preserves the required left extension in this case since this left extension amounts to the description of each object of A as an α-filtered colimit of α-presentable objects, and T and $A(i_0, 1)$ both preserve α-filtered colimits [Adamek-Rosický, 1994].

Writing $U : T\text{-Alg} \rightarrow A$ for the forgetful functor and F for its left adjoint, factor the composite Fi_0 as

$$\begin{tikzcd}
\Theta_0 \arrow[r, "i_0"] \arrow[dr, "j"'] & A \arrow[r, "F"] \arrow[d, phantom, "="] & T\text{-Alg} \\
& \Theta_T \arrow[ur, "i"'] &
\end{tikzcd}$$

an identity on objects functor j followed by a fully faithful functor i to define the category Θ_T.

4.4. Definition. *Let* (T, Θ_0) *be a monad with arities on* A. *Then the* nerve functor *for* (T, Θ_0) *is*

$$T\text{-Alg}(i, 1) : T\text{-Alg} \rightarrow \widehat{\Theta}_T$$

and so for a T*-algebra* X *we call* $T\text{-Alg}(i, X) \in \widehat{\Theta}_T$ *the* nerve of X. *For* $Z \in \widehat{\Theta}_T$ *satisfies the* Segal condition *for* (T, Θ_0) *when* $\mathrm{res}_j Z$ *is in the image of*

$$A(i_0, 1) : A \rightarrow \widehat{\Theta}_0.$$

The main purpose of this section is to characterise the nerves of T-algebras as those presheaves satisfying the Segal condition, and to explain the important instances of this result. For a given choice of unit $\overline{\eta}$ and counit $\overline{\varepsilon}$ of $\mathrm{lan}_j \dashv \mathrm{res}_j$, we denote by $(\overline{T}, \overline{\eta}, \overline{\mu})$ the monad on $\widehat{\Theta}_0$ which arises from this adjunction. First we note that $\overline{T}$-Alg may be identified with $\widehat{\Theta}_T$.

4.5. Lemma. $\mathrm{res}_j : \widehat{\Theta} \rightarrow \widehat{\Theta}_0$ *is monadic.*

Proof. Let $f_1, f_2 : X_1 \rightrightarrows X_2$ be maps in $\widehat{\Theta}_T$. Since coequalisers are formed componentwise in presheaf categories, any coequaliser of $f_1 j^{\mathrm{op}}$ and $f_2 j^{\mathrm{op}}$ lifts to a unique coequaliser of f_1 and f_2. Thus by the absolute coequaliser form of the Beck theorem [Paré, 1969] the comparison functor $\widehat{\Theta}_T \to \overline{T}\text{-Alg}$ is an isomorphism. ∎

We shall now induce canonical isomorphisms

$$\begin{tikzcd}
T\text{-Alg} \arrow[r, "T\text{-Alg}(i,1)"] \arrow[d, "U"'] \arrow[dr, phantom, "\overset{\kappa}{\Longrightarrow}"] & \widehat{\Theta}_T \arrow[d, "\mathrm{res}_j"] \\
A \arrow[r, "A(i_0,1)"'] & \widehat{\Theta}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "A(i_0,1)"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & \widehat{\Theta}_0 \arrow[d, "\mathrm{lan}_j"] \\
T\text{-Alg} \arrow[r, "T\text{-Alg}(i,1)"'] & \widehat{\Theta}_T
\end{tikzcd}$$

which when pasted together provide a monad functor from T to $\overline{T}$. We define κ as the unique 2-cell such that

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[dr, "1"] & T\text{-Alg} \arrow[r, "T\text{-Alg}(i,1)"] \arrow[d, "U"] \arrow[dr, phantom, "\overset{\kappa}{\Longrightarrow}"] \arrow[l, phantom, "\overset{\eta}{\Longrightarrow}" near start, shift right=4] & \widehat{\Theta}_T \arrow[d, "\mathrm{res}_j"] \\
\Theta_0 \arrow[u, "i_0"] \arrow[r, "i_0"'] \arrow[ur, phantom, "\Uparrow\mathrm{id}" near start] \arrow[rr, bend right=50, "y"'] & A \arrow[r, "A(i_0,1)"'] \arrow[d, phantom, "\chi^{i_0}\Uparrow"] & \widehat{\Theta}_0 \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
\Theta_0 \arrow[r, "j"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\chi^{yj}}{\Longrightarrow}"] & \Theta_T \arrow[r, "i"] \arrow[d, "y"] & T\text{-Alg} \arrow[dl, "T\text{-Alg}(i,1)"] \arrow[dl, phantom, "\overset{\chi^i}{\Longrightarrow}" near end, shift left=3] \\
\widehat{\Theta}_0 & \widehat{\Theta}_T \arrow[l, "\mathrm{res}_j"] &
\end{tikzcd}
\qquad (2)$$

Note that χ^{i_0} exhibits $A(i_0, 1)$ as a left extension of y along i_0, and since η as the unit of an adjunction is an absolute left extension, we have by the composability of left extensions that $A(i_0, U)$ is exhibited as a left extension of y along Fi_0 on the left hand side of (2). Thus the above definition of κ makes sense since $Fi_0 = ij$. In addition to this we have

4.6. **Lemma.** *The 2-cell κ defined by (2) is an isomorphism and exhibits* res_j *as a left extension of $A(i_0, U)$ along* T-Alg$(i, 1)$.

Proof. The right hand side of (2) exhibits $\mathrm{res}_j T\text{-Alg}(i, 1)$ as a left extension of y along ij by proposition(3.4) of [Weber, 2007], and so κ is an isomorphism. Since i is fully faithful χ^i is invertible. Since χ^{yj} exhibits res_j as a left extension, it follows that the right hand side of (2) exhibits res_j as a left extension of y along $T\text{-Alg}(i, ij)$ and so by the composability of left extensions, κ exhibits res_j as a left extension of $A(i_0, U)$ along $T\text{-Alg}(i, 1)$. ∎

Now we define ψ to be the unique 2-cell such that

$$\begin{tikzcd}
\Theta_0 \arrow[r, "i_0"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Longrightarrow}"] & A \arrow[r, "A(i_0,1)"] \arrow[d, "F"] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & \widehat{\Theta}_0 \arrow[d, "\mathrm{lan}_j"] \\
\Theta_T \arrow[r, "i"'] \arrow[rr, bend right=50, "y"'] & T\text{-Alg} \arrow[r, "T\text{-Alg}(i,1)"'] \arrow[d, phantom, "\chi^i\Uparrow"] & \widehat{\Theta}_T \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
\Theta_0 \arrow[r, "i_0"] \arrow[d, "j"'] \arrow[dr, "y"] & A \arrow[d, "A(i_0,1)"] \arrow[dl, phantom, "\overset{\chi^{i_0}}{\Rightarrow}" near start] \\
\Theta_T \arrow[r, phantom, "\overset{\phi_j}{\Longrightarrow}"] \arrow[dr, "y"'] & \widehat{\Theta}_0 \arrow[d, "\mathrm{lan}_j"] \\
& \widehat{\Theta}_T
\end{tikzcd}
\qquad (3)$$

Since F is a left adjoint and i_0 is dense the identity cell on the left hand side of (3) exhibits F as a left extension of ij along i_0. By the isomorphism κ and from the condition that i_0 endows T with arities, this left extension is preserved by $\mathrm{res}_j T\text{-}\mathrm{Alg}(i,1)$. However res_j creates colimits, and so this left extension is in fact preserved by $T\text{-}\mathrm{Alg}(i,1)$. Thus since χ^i is an isomorphism, $T\text{-}\mathrm{Alg}(i,F)$ is exhibited as a left extension of $y_{\Theta_T}j$ along i_0 on the left hand side of (3), and so the above definition of ψ does indeed make sense. In addition to this we have

4.7. Lemma. *The 2-cell ψ defined by (3) is an isomorphism and exhibits* lan_j *as a left extension of* $T\text{-}\mathrm{Alg}(i,F)$ *along* $A(i_0,1)$.

Proof. Since χ^{i_0} is an isomorphism the right hand side of (3) exhibits lan_j as a left extension of $y_{\Theta_T}j$ along $A(i_0,i_0)$, and so by the composability of left extensions ψ exhibits lan_j as a left extension of $T\text{-}\mathrm{Alg}(i,F)$ along $A(i_0,1)$. The 2-cell ϕ_j is invertible and lan_j is a left adjoint and so the composite cell of (3) exhibits $\mathrm{lan}_j A(i_0,1)$ as a left extension of $y_{\Theta_T}j$ along i_0, and so ψ is an isomorphism. ∎

Following [Lack-Street, 2002] we shall refer to a monad in a 2-category $\mathcal{K}$ as a pair (A,s) where $s : A \to A$ is the underlying endoarrow of the monad, and the unit and multiplication are left unmentioned. Recall from [Street, 1972] [Lack-Street, 2002] that a *monad morphism* $(A,s) \to (B,t)$ consists of a pair (f,ϕ), where $f : A \to B$ and $\phi : tf \to fs$, which are compatible with the monad structures in the sense that

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & {} \arrow[d] \arrow[d, bend left=70, "1"] \arrow[d, phantom, shift left=4, "\overset{\eta}{\Leftarrow}"] \\
{} \arrow[r, "f"'] & {}
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[d, bend right=70, "s"'] \arrow[d, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"] \arrow[r, "f"] & {} \arrow[d, "1"] \arrow[dl, phantom, "="] \\
{} \arrow[r, "f"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[dd, "s"'] \arrow[ddr, phantom, "\overset{\phi}{\Leftarrow}"] & {} \arrow[dd] \arrow[dr, "t"] & \\
& \arrow[l, phantom, "\overset{\mu}{\Leftarrow}"] & {} \arrow[dl, "t"] \\
{} \arrow[r, "f"'] & {} &
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[rr, "f"] \arrow[dd, "s"'] \arrow[dr] & & {} \arrow[dr, "t"] & \\
{} \arrow[r, phantom, "\overset{\mu}{\Leftarrow}"] & {} \arrow[dl] \arrow[rr] \arrow[ur, phantom, "\overset{\phi}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & & {} \arrow[dl, "t"] \\
{} \arrow[rr, "f"'] & & {} &
\end{tikzcd}$$

Defining φ as the inverse of the composite

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "A(i_0{,}1)"'] \arrow[dr, phantom, "\overset{\psi}{\Leftarrow}"] & T\text{-}\mathrm{Alg} \arrow[r, "U"] \arrow[d, "T\text{-}\mathrm{Alg}(i{,}1)"] \arrow[dr, phantom, "\overset{\kappa}{\Leftarrow}"] & A \arrow[d, "A(i_0{,}1)"] \\
\widehat{\Theta}_0 \arrow[r, "\mathrm{lan}_j"'] & \widehat{\Theta}_T \arrow[r, "\mathrm{res}_j"'] & \widehat{\Theta}_0
\end{tikzcd}$$

and recalling that the monad structure of $\overline{T}$ is determined by a choice of unit $\overline{\eta}$ and counit $\overline{\varepsilon}$ of $\mathrm{lan}_j \dashv \mathrm{res}_j$, we have

4.8. Proposition. *One can choose the unit $\overline{\eta}$ and counit $\overline{\varepsilon}$ of* $\mathrm{lan}_j \dashv \mathrm{res}_j$ *in a unique way so that $(A(i_0,1),\varphi)$ is a monad morphism from T to $\overline{T}$.*

Proof. For any 2-category $\mathcal{K}$ one may associate another 2-category $\mathcal{C}(\mathcal{K})$ as follows. The objects of $\mathcal{C}(\mathcal{K})$ are the arrows of $\mathcal{K}$. Given arrows f and g of $\mathcal{K}$, an arrow $f \rightarrow g$ consists of (h, k, ϕ) as in

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & {} \arrow[d, "k"] \\
{} \arrow[r, "g"'] & {}
\end{tikzcd}$$

where ϕ is invertible and exhibits k as a left extension of gh along f, and k is a left adjoint. Composition of arrows is given by vertical pasting and is well-defined since left adjoints preserve left extensions. Given arrows (h_1, k_1, ϕ_1) and (h_2, k_2, ϕ_2) from f to g in $\mathcal{C}(\mathcal{K})$, a 2-cell consists of $\alpha : h_1 \rightarrow h_2$ and $\beta : k_1 \rightarrow k_2$ such that

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, bend right=50] \arrow[d, bend left=50] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\phi_2}{\Rightarrow}"] & {} \arrow[d, bend left=50] & & {} \arrow[r] \arrow[d, bend right=50] \arrow[dr, phantom, "\overset{\phi_1}{\Rightarrow}"] & {} \arrow[d, bend right=50] \arrow[d, bend left=50] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\
{} \arrow[r] & {} \arrow[r, phantom, "="] & {} & {} \arrow[r] & {}
\end{tikzcd}$$

By the elementary properties of left extensions the 2-functor $\mathrm{dom} : \mathcal{C}(\mathcal{K}) \rightarrow \mathcal{K}$ whose object map takes the domain of an arrow is locally fully faithful[13]. Thus given arrows $(l, \overline{l}, \phi_1)$ and $(r, \overline{r}, \phi_2)$ in $\mathcal{C}(\mathcal{K})$, and an adjunction $l \dashv r$ in $\mathcal{K}$, one obtains a unique adjunction $\overline{l} \dashv \overline{r}$ compatible with ϕ_1 and ϕ_2. By lemmas(4.6) and (4.7) one can apply this last observation to κ and ψ from which the result follows by the definition of φ. ∎

Before moving on to the proof of our most general nerve characterisation, we describe a general lemma on monad morphisms that applies to $(A(i_0, 1), \varphi)$. First recall that given a monad (A, s) in a 2-category $\mathcal{K}$, that an *algebra* for s consists of $x : X \rightarrow A$ and $a : sx \rightarrow x$ such that $a(\eta x) = \mathrm{id}$ and $a(\mu x) = a(sa)$. For example the Eilenberg-Moore object of s consists of $u : A^s \rightarrow A$ and an s-algebra $\sigma : su \rightarrow u$ which satisfies a universal property. Now for any monad morphism $(f, \phi) : (A, s) \rightarrow (B, t)$ in $\mathcal{K}$ the assignment

$$sx \xrightarrow{a} x \qquad \mapsto \qquad tfx \xrightarrow{\phi x} fsx \xrightarrow{fa} fx$$

sends s-algebra structures on x to t-algebra structures on fx, and the following observation is immediate.

4.9. Lemma. *Let (f, ϕ) be a monad morphism $(A, s) \rightarrow (B, t)$ in a 2-category $\mathcal{K}$ such that f is fully-faithful and ϕ is an isomorphism. Then the above assignment gives a bijection between s-algebra structures on x and t-algebra structures on fx.*

4.10. Theorem. *Let (T, Θ_0) be a monad with arities on A.*

1. *The nerve functor $T\text{-}\mathrm{Alg}(i, 1)$ is fully faithful.*

2. *$Z \in \widehat{\Theta}_T$ is the nerve of a T-algebra iff it satisfies the Segal condition.*

---

[13] Meaning that dom's hom-functors are fully faithful.

Proof. We denote the pullback and pseudo pullback of $A(i_0, 1)$ and res_j as

$$\begin{tikzcd} P_1 \arrow[r, "q_1"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\Theta}_T \arrow[d, "\mathrm{res}_j"] \\ A \arrow[r, "A(i_0,1)"'] & \widehat{\Theta}_0 \end{tikzcd} \qquad \begin{tikzcd} P_2 \arrow[r, "q_2"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & \widehat{\Theta}_T \arrow[d, "\mathrm{res}_j"] \\ A \arrow[r, "A(i_0,1)"'] & \widehat{\Theta}_0 \end{tikzcd}$$

and write $k_1 : P_1 {\to} P_2$ and $k_2 : T\text{-Alg}{\to}P_2$ for the canonical maps induced by the universal property of P_2: $k_1\lambda = \mathrm{id}$ and $k_2\lambda = \kappa$. We have

$$\begin{tikzcd} T\text{-Alg} \arrow[r, "k_2"] \arrow[d, "T\text{-Alg}(i,1)"'] \arrow[dr, phantom, "=", near start] & P_2 \arrow[dl, "q_2" description] \\ \widehat{\Theta}_T & P_1 \arrow[l, "q_1"] \arrow[u, "k_1"'] \arrow[ul, phantom, "=", near start] \end{tikzcd}$$

and k_1 is an equivalence by examples(3.9) and (3.12). Since fully faithful maps are pullback stable q_1 is fully faithful since $A(i_0, 1)$ is. Thus by the above diagram q_2 is fully faithful. If k_2 is an equivalence then T-Alg$(i, 1)$ is also fully faithful by the above diagram, and the essential images[14] of q_1 and T-Alg$(i, 1)$ are the same. Since the essential image of q_2 consists of those $Z \in \widehat{\Theta}_T$ satisfying the Segal condition by definition, to finish the proof it suffices to show that k_2 is an equivalence.

Now $\mathrm{res}_j q_2$ is a $\overline{T}$-algebra with structure map $\mathrm{res}_j\overline{\varepsilon}q_2$, and so by example(3.9) we can define $a_1 : \overline{T}A(i_0, p_2){\to}A(i_0, p_2)$ as the unique $\overline{T}$-algebra structure on $A(i_0, p_2)$ making λ an isomorphism of $\overline{T}$-algebras. By lemma(4.9) we can define $a_2 : Tp_2{\to}p_2$ as the unique T-algebra structure on p_2 such that $A(i_0, a_2){\circ}\varphi p_2 = a_1$. By the universal property of $U\varepsilon : TU{\to}U$ as the Eilenberg-Moore object of T, there is a unique $k_3 : P_2{\to}T$-Alg such that $Uk_3 = p_2$ and $U\varepsilon k_3 = a_2$. We will now verify that k_3 is a pseudo inverse for k_2. We have $p_2k_2k_3 = p_2$ and so to obtain $k_2k_3{\cong}1$ it suffices, by the 2-dimensional of the universal property of P_2, to give an isomorphism $\phi : q_2{\to}q_2k_2k_3$ such that $\mathrm{res}_j\phi{\circ}\lambda = \lambda k_2k_3$. But this is the same as giving a $\overline{T}$-algebra isomorphism $\phi_2 : \mathrm{res}_jq_2{\to}\mathrm{res}_jq_2k_2k_3$ such that $\phi_2{\circ}\lambda = \lambda k_2k_3$. There is one such isomorphism because λ and λk_2k_3 are themselves $\overline{T}$-algebra isomorphisms. To give an isomorphism $1{\cong}k_3k_2$ is the same as giving an isomorphism $U{\cong}Uk_3k_2$ of T-algebras. We have $Uk_3k_2 = U$ and so it suffices to see that this is an identity of T-algebras, that is, that $U\varepsilon = U\varepsilon k_3k_2$. Since $A(i_0, 1)$ is fully faithful and by the definition of k_3, this is the same as showing that $A(i_0, U\varepsilon) = A(i_0, a_2k_2)$. From the definitions of the various 2-cells involved it is straight forward to verify that

$$\begin{aligned} \lambda k_2 \circ A(i_0, U\varepsilon) \circ \varphi U &= \mathrm{res}_j(T\text{-Alg}(i, 1)\varepsilon \circ \psi^{-1}U) \\ \lambda k_2 \circ A(i_0, a_2k_2) \circ \varphi U &= \mathrm{res}_j(\overline{\varepsilon}T\text{-Alg}(i, 1) \circ \mathrm{lan}_j\kappa) \end{aligned}$$

and so it suffices to show that $T\text{-Alg}(i, 1)\varepsilon \circ \psi^{-1}U = \overline{\varepsilon}T\text{-Alg}(i, 1) \circ \mathrm{lan}_j\kappa$, and this follows from the definition of $\overline{\varepsilon}$ via proposition(4.8). ■

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[14] The *essential image* of a fully faithful functor $f : A{\to}B$ is the full subcategory of B consisting of the $b \in B$ such that $fa{\cong}b$ for some a in A.

We will now begin to specialise this result and discuss examples. First we assume that A is a presheaf category: $A = \widehat{\mathbb{C}}$ for some small category $\mathbb{C}$, and that i_0 is the inclusion of a full subcategory which includes the representables. Thus we have

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "j_0"] \arrow[dr, "y_{\mathbb{C}}"'] & {} \arrow[d, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & \Theta_0 \arrow[dl, "i_0"] \\
& \widehat{\mathbb{C}} &
\end{tikzcd} \tag{4}$$

where i_0 and j_0 are the inclusions. For any $p \in \Theta_0$ we write

$$\begin{tikzcd}
y/p \arrow[r] \arrow[d, "\pi_p"'] \arrow[dr, phantom, "\overset{\overline{p}}{\Rightarrow}"] & 1 \arrow[d, "p"] \\
\mathbb{C} \arrow[r, "y"'] & \widehat{\mathbb{C}}
\end{tikzcd}$$

for the canonical lax pullback square, and recall that this cocone exhibits p as a colimit of representables. Explicitly, the component of $\overline{p}$ corresponding to $x : C \to p$ in y/p is just x as an arrow of $\widehat{\mathbb{C}}$. By definition the cocone $\overline{p}$ lives in Θ_0, and so one can consider the cocone $j\overline{p}$ in Θ_T.

We shall now show that $Z \in \widehat{\Theta}_T$ satisfies the Segal condition iff it sends each cocone $j\overline{p}$ to a limit cone, by characterising the image of $\widehat{\mathbb{C}}(i_0, 1)$. Since i_0 is fully faithful (4) exhibits j_0 as an absolute left lifting of $y_{\mathbb{C}}$ along i_0, and so (4) exhibits i_0 as a pointwise left extension of y along j_0 by axiom(2) of the good yoneda structure for CAT. This implies $i_0 \cong \Theta_0(j_0, 1)$ so that $\mathrm{res}_{j_0} \dashv \widehat{\mathbb{C}}(i_0, 1)$. Thus we have

4.11. LEMMA. *$\widehat{\mathbb{C}}(i_0, 1)$ is fully-faithful and has left adjoint res_{j_0}.*

Write $\tilde{\eta}$ for the unit of $\mathrm{res}_{j_0} \dashv \widehat{\mathbb{C}}(i_0, 1)$. We want to characterise the image of $\widehat{\mathbb{C}}(i_0, 1)$, and by lemma(4.11), $X \in \widehat{\Theta}_0$ is in this image iff $\tilde{\eta}_X$ is invertible. Thus we want a useful explicit description of $\tilde{\eta}_X$. For $p \in \Theta_0$ we must describe a function

$$\tilde{\eta}_{X,p} : Xp \to \widehat{\mathbb{C}}(p, Xj_0^{\mathrm{op}})$$

and this definition must be natural in X and p. Since $\overline{p}$ is a colimit cocone, the functions

$$\widehat{\mathbb{C}}(x, Xj_o^{\mathrm{op}}) : \widehat{\mathbb{C}}(p, Xj_o^{\mathrm{op}}) \to \widehat{\mathbb{C}}(C, Xj_o^{\mathrm{op}})$$

for all $x : C \to p$ in y/p, form a limit cone in Set. Thus we can define $\tilde{\eta}_{X,p}$ to be the unique function such that for all $x : C \to p$ the square

$$\begin{tikzcd}
Xp \arrow[r, "X(x)"] \arrow[d, "\tilde{\eta}_{X,p}"'] & XC \arrow[d, "\iota_{C,X}"] \\
\widehat{\mathbb{C}}(p, Xj_o^{\mathrm{op}}) \arrow[r, "\widehat{\mathbb{C}}(x, Xj_o^{\mathrm{op}})"'] & \widehat{\mathbb{C}}(C, Xj_o^{\mathrm{op}})
\end{tikzcd}$$

commutes, where ι is the induced isomorphism in

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "y"] \arrow[dr, "y"'] & \arrow[d, phantom, "\overset{\chi^y}{\Rightarrow}"] & \widehat{\mathbb{C}} \arrow[dl, "\widehat{\mathbb{C}}(y{,}1)"] & = & \mathbb{C} \arrow[rr, "y"] \arrow[dr, "y"'] & \arrow[d, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & \widehat{\mathbb{C}} \arrow[dl, "1"', ""{name=A}] \arrow[dl, bend left=60, "\widehat{\mathbb{C}}(y{,}1)", ""'{name=B}] \arrow[Rightarrow, from=A, to=B, "\iota"] \\
& \widehat{\mathbb{C}} & & & & \widehat{\mathbb{C}} &
\end{tikzcd}$$

4.12. Lemma. *The 2-cell $\tilde{\eta}$ just defined is a unit for* $\mathrm{res}_{j_0} \dashv \widehat{\mathbb{C}}(i_0, 1)$.

Proof. We already know there exists such an adjunction and that $\widehat{\mathbb{C}}(i_0, 1)$ is fully faithful. Thus we have $\mathrm{res}_{j_0}\widehat{\mathbb{C}}(i_0, 1) \cong 1_{\widehat{\mathbb{C}}}$ and so by [Weber, 2007] lemma(2.6) it suffices to show that $\mathrm{res}_{j_0}\tilde{\eta}$ and $\tilde{\eta}\widehat{\mathbb{C}}(i_0, 1)$ are invertible. This in turn amounts to saying that $\tilde{\eta}_{X,p}$ is invertible when p is a representable (ie in $\mathbb{C}$) or when X is of the form $\widehat{\mathbb{C}}(i_0, Z)$ for some $Z \in \widehat{\mathbb{C}}$. In each case $\tilde{\eta}_{X,p}$ turns out, by definition, to be a component of ι. ■

From lemmas(4.11) and (4.12) the following characterisation of the image of $\widehat{\mathbb{C}}(i_0, 1)$ is now immediate.

4.13. Proposition. *$X \in \widehat{\Theta}_0$ is in the image of $\widehat{\mathbb{C}}(i_0, 1)$ iff for each $p \in \Theta_0$, X sends the colimit cocone $\overline{p}$ to a limit cone.*

4.14. Corollary. *[Leinster, 2004] Let (T, Θ_0) be a monad with arities on $\widehat{\mathbb{C}}$ and i_0 be the inclusion of a full subcategory that contains the representables. Then $Z \in \widehat{\Theta}_T$ satisfies the Segal condition iff for all $p \in \Theta_0$ it sends the cocone $j\overline{p}$ to a limit cone.*

4.15. Example. One can apply theorem(4.10) and corollary(4.14) in the case where $\mathbb{C} = 1$ (so $\widehat{\mathbb{C}} = \mathrm{Set}$) and T is a finitary monad on Set seen as having arities via the inclusion of a skeleton of the category of finite sets. In this case Θ_T^{op} is precisely the Lawvere theory associated to T. Recall that a *model* of the theory Θ_T^{op} in a category $\mathcal{E}$ with products is a finite product preserving functor $\Theta_T^{\mathrm{op}} \to \mathcal{E}$. Theorem(4.10) gives the well-known identification between the algebras of T and the models of its associated Lawvere theory in Set: to say that $Z \in \widehat{\Theta}_T$ satisfies the Segal condition is the same as saying that $Z : \Theta_T^{\mathrm{op}} \to \mathrm{Set}$ preserves finite products.

We now consider the case of a p.r.a monad (T, η, μ) on a presheaf category $\widehat{\mathbb{C}}$, and explain how such a monad always comes equipped with arities. So we shall identify the objects of Θ_0 in the next definition, and then proceed to explain how they provide arities for T.

4.16. Definition. *Let $\mathbb{C}$ be a small category and (T, η, μ) be a p.r.a monad on $\widehat{\mathbb{C}}$. We define Θ_0 to be the full subcategory of $\widehat{\mathbb{C}}$ consisting of those objects in the image of $E_T : y_{\mathbb{C}}/T1 \to \widehat{\mathbb{C}}$ (see proposition(2.10) and remark(2.12)). An object $p \in \widehat{\mathbb{C}}$ is a T-*cardinal *when it is isomorphic to an object of Θ_0.*

4.17. Example. Let T be the category monad on Graph. The representables in Graph are the graphs $[0]$ and $[1]$. Trivially $[0]{\rightarrow}TX$ is T-generic iff $X{\cong}[0]$. Thus by example(2.8), T-cardinals are those graphs of the form $[n]$. Moreover the category Θ_T in this case is the category Δ of non-empty ordinals and order-preserving maps.

4.18. Example. Let T be the strict ω-category monad on the category of globular sets. By [Weber, 2004] proposition(9.1) T-cardinals are exactly the globular cardinals of [Batanin, 1998] [Street, 2000]. Moreover the category Θ_T in this case is Joyal's category Θ [Joyal, 1997].

4.19. Example. Let T be the symmetric multicategory monad on $\widehat{\mathbb{M}}$, the category of multigraphs, described in example(2.14). Then the category Ω as defined in [Moerdijk-Weiss, 2007a] section(3) is equivalent to Θ_T. As explained in example(2.14) Θ_0 is not skeletal, although its isomorphism classes are in bijection with planar trees. In [Moerdijk-Weiss, 2007a], the category Ω is defined to have non-planar trees as objects, and the objects of $\widehat{\Omega}$ are called *dendroidal sets.*

Note that given a p.r.a monad T on $\widehat{\mathbb{C}}$, an object $X \in \widehat{\mathbb{C}}$ is a T-cardinal iff there exists $C \in \mathbb{C}$ a generic morphism $C{\rightarrow}TX$, by remark(2.11).

4.20. Proposition. *Let T be a p.r.a monad on $\widehat{\mathbb{C}}$.*

1. *Representables are T-cardinals.*
2. *If p is a T-cardinal and $g : p{\rightarrow}Tq$ is T-generic then q is a T-cardinal.*

Proof. (1): for $C \in \mathbb{C}$ we shall see that $\eta_C : C{\rightarrow}TC$ is T-generic. From

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\eta_C"'] \arrow[dr, phantom, "="] & TX \arrow[d, "T\gamma"] \\
TC \arrow[r, "T\beta"'] & TZ
\end{tikzcd}$$

one obtains the desired T-filler δ as follows:

$$\begin{tikzcd}
C \arrow[rr, "\alpha"] \arrow[dr, dotted, "\delta"] \arrow[dddd, "\eta"'] \arrow[ddr, bend right, "\beta"'] & = & TX \arrow[dddd, "T\gamma"] \\
= & X \arrow[ur, "\eta"'] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \\
& Z \arrow[ddr, "\eta"] & \\
= & & \\
TC \arrow[rr, "T\beta"'] & & TZ
\end{tikzcd}$$

(2): since p is a T-cardinal we have a T-generic $g_1 : C{\rightarrow}Tp$, thus the composite $T(g)g_1$ is T^2-generic by [Weber, 2004] lemma 5.14. Since μ is cartesian $\mu_q T(g)g_1$ is T-generic, by [Weber, 2004] proposition 5.10, and so exhibits q as a T-cardinal. ∎

4.21. Examples. By proposition(4.20) one obtains a "generic-free" factorisation system on the full sub-category of $\mathrm{Kl}(T)$ given by the T-cardinals. In the case where T is the free category monad, this sub-category is the category of non-empty ordinals and order preserving maps. The factorisation system is given by the factorisation of monotone maps $[n]\to[m]$ into a map which preserves top and bottom elements (the T-generics), followed by a monotone injection which preserves consecutive elements (the free maps). The analogous factorisation for the strict ω-category monad was described in [Berger, 2002].

4.22. Proposition. *Let $\mathbb{C}$ be a small category and (T, η, μ) be a p.r.a monad on $\widehat{\mathbb{C}}$. Then the inclusion $i_0 : \Theta_0\to\widehat{\mathbb{C}}$ of the T-cardinals provides T with arities.*

Proof. We have

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "j_0"] \arrow[dr, "y_{\mathbb{C}}"'] \arrow[dr, phantom, bend left=20, "\overset{\mathrm{id}}{\Rightarrow}"] & \Theta_0 \arrow[r, "i_0"] \arrow[d, "i_0"] & \widehat{\mathbb{C}} \arrow[dl, "1"] \arrow[dl, phantom, bend right=20, "\overset{\mathrm{id}}{\Rightarrow}"'] \\
& \widehat{\mathbb{C}} &
\end{tikzcd}$$

and we saw above that the left most 2-cell here is a left extension. The composite 2-cell is also a left extension since $y_{\mathbb{C}}$ is dense, and so the right most 2-cell is a left extension also, and so i_0 is dense. To finish the proof we must show that the functor $\widehat{\mathbb{C}}(i_0, T)$ preserves the left extension

$$\begin{tikzcd}
\Theta_0 \arrow[rr, "i_0"] \arrow[dr, "i_0"'] & \arrow[d, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & \widehat{\mathbb{C}} \arrow[dl, "1"] \\
& \widehat{\mathbb{C}} &
\end{tikzcd}$$

For $X \in \widehat{\mathbb{C}}$ the lax pullback

$$\begin{tikzcd}
i_0/X \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & 1 \arrow[d, "X"] \\
\Theta_0 \arrow[r, "i_0"'] & \widehat{\mathbb{C}}
\end{tikzcd}$$

exhibits X as a colimit of T-cardinals by the density of i_0. We must show that this colimit is preserved by $\widehat{\mathbb{C}}(i_0, T)$. Given

$$\begin{tikzcd}
i_0/X \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & 1 \arrow[d, "Z"] \\
\Theta_0 \arrow[r, "\widehat{\mathbb{C}}(i_0, Ti_0)"'] & \widehat{\Theta}_0
\end{tikzcd}$$

we define $\overline{\phi} : \widehat{\mathbb{C}}(i_0, TX)\to W$ by $\overline{\phi}_p(f) = \phi_{h_f,p}(g_f)$ for $f : p\to TX$. To see that this is natural in p let $k : q\to p$ be in Θ_0. Inducing δ as the unique T-fill in the commutative diagram on the left we obtain the commutative diagram on the right

$$\begin{tikzcd}
q \arrow[r, "k"] \arrow[d, "g_{fk}"'] & p \arrow[r, "f"] \arrow[d, "g_f"] & TX \\
TD_{fk} \arrow[r, dotted, "T\delta"] \arrow[urr, bend right=60, "Th_{fk}"'] & TD_f \arrow[ur, "Th_f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathbb{C}}(q, TD_f) \arrow[drr, "\phi_{h_f,q}"] & & \widehat{\mathbb{C}}(q, TD_{fk}) \arrow[ll, "\widehat{\mathbb{C}}(q,T\delta)"'] \arrow[d, "\phi_{h_{fk},q}"] \\
\widehat{\mathbb{C}}(p, TD_f) \arrow[u, "\widehat{\mathbb{C}}(k,TD_f)"] \arrow[d, "\phi_{h_f,p}"'] & & Zq \\
Zp \arrow[urr, "Zk"'] & &
\end{tikzcd}$$

and so by

$$\begin{aligned}
W(k)\overline{\phi}_p(f) &= \phi_{h_f,p}(g_f) = W(k)\phi_{h_f,p}(g_f) = \phi_{h_f,q}(g_f k) \\
&= \phi_{h_f,q}(T(\delta)g_{fk}) = \phi_{h_{fk},q}(g_{fk}) = \overline{\phi}_q(fk)
\end{aligned}$$

$\overline{\phi}_p$ is indeed natural in p. The equation

$$\begin{tikzcd}
i_0/X \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & 1 \arrow[d, "Z"] \\
\Theta_0 \arrow[r, "\widehat{\mathbb{C}}(i_0,Ti_0)"'] & \widehat{\Theta}_0
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
i_0/X \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & 1 \arrow[d] \arrow[d, bend left=80, "Z"] \arrow[d, phantom, shift left=5ex, "\overset{\overline{\phi}}{\Rightarrow}"] \\
\Theta_0 \arrow[r, "\widehat{\mathbb{C}}(i_0,Ti_0)"'] & \widehat{\Theta}_0
\end{tikzcd} \tag{5}$$

says that for each $h : p \to X$ and $q \in \Theta_0$, that $\phi_{h,q}(f) = \overline{\phi}_q \widehat{\mathbb{C}}(q, Th)$, and this follows by the definition of $\overline{\phi}$ and the naturality of ϕ. On the other hand for $f : p \to TX$, taking $h = h_f$ we obtain $\overline{\phi}_p(f) = \phi_{h_f,p}(g_f)$ from this equation and so $\overline{\phi}$ is unique satisfying (5). ∎

4.23. Example. Applying proposition(4.22) and theorem(4.10) to the category monad on Graph gives the well-known characterisation of nerves of categories via the Segal condition.

4.24. Example. Applying proposition(4.22) and theorem(4.10) to the strict ω-category monad on the category of globular sets gives the characterisation of nerves of strict ω-categories given by [Berger, 2002] theorem(1.12).

4.25. Remark. In the above analysis of a p.r.a monad T on $\widehat{\mathbb{C}}$ one can replace Θ_0 by any full subcategory Θ'_0 of $\widehat{\mathbb{C}}$ satisfying the following conditions:

1. Θ'_0 contains the representables,
2. if $p \in \Theta'_0$ and $g : p \to Tq$ is T-generic then $q \in \Theta'_0$, and
3. Θ'_0 is equivalent to a small category.

Thus it follows that given a cartesian monad morphism $\phi : S \to T$, the inclusion $i_0 : \Theta_0 \to \widehat{\mathbb{C}}$ also endows S with arities. To see this one need only check (2) for (S, Θ_0), and this follows from the corresponding property for (T, Θ_0) because composition with components of ϕ preserves generics by [Weber, 2004] proposition(5.10).

4.26. Example. In [Berger, 2002] theorem(1.17) nerves of algebras of ω-operads are characterised. An ω-operad is a cartesian monad morphism $S \to T$ where T is the strict ω-category monad on the category of globular sets. In view of remark(4.25) one recaptures this result by applying theorem(4.10) and corollary(4.14) to (S, Θ_0), where Θ_0 is the full subcategory consisting of the T-cardinals. Remember that Θ_S is defined from Θ_0 via the identity on objects fully faithful factorisation of Fi_0, and this Θ_S is what in [Berger, 2002] is called the *globular theory* of the associated ω-operad. Note that one gets another nerve characterisation of S-algebras by using the S-cardinals instead of the T-cardinals.

4.27. Example. In [Moerdijk-Weiss, 2007b] nerves of operads are characterised in proposition(5.3) and theorem(6.1). In view of examples(2.14) and (4.19), one obtains these results by applying theorem(4.10), corollary(4.14) and proposition(4.20) to the symmetric multicategory monad on the category $\widehat{\mathbb{M}}$ of multigraphs.

5. Familial 2-functors

In this section we introduce familial 2-functors and then discuss in detail the paradigmatic example Fam.

5.1. The definition of familial 2-functor. The notion of familial 2-functor is intended to be a 2-categorical analogue of p.r.a-ness. The notion of p.r.a has an obvious *naive* 2-categorical analogue because for a 2-category $\mathcal{A}$ and $X \in \mathcal{A}$, one has the naive slice 2-category $\mathcal{A}/X$, and for a 2-functor $T : \mathcal{A} \to \mathcal{B}$, the effect of T on arrows into X is a 2-functor $T_X : \mathcal{A}/X \to \mathcal{B}/TX$. Thus one can just imitate definition(2.3) at this level to provide the definitions of p.r.a 2-functor and p.r.a 2-monad. Generic morphisms work in much the same way, except that in order to obtain the 2-categorical analogue of proposition(2.6), one must describe a 2-dimensional aspect. That is, a one-cell $f : B \to TA$ is *T-generic* when it is T-generic in the 1-categorical sense as defined in section(2) and, given

$$\begin{tikzcd}
B \arrow[r, bend left, "\alpha_2"] \arrow[r, bend right, "\alpha_1"'] \arrow[r, phantom, "\phi \Uparrow"] \arrow[d, "f"'] & TX \arrow[d, "T\gamma"] \\
TA \arrow[r, "T\beta"'] & TZ
\end{tikzcd}$$

such that $T(\beta)f = T(\gamma)\alpha_1 = T(\gamma)\alpha_2$ and $T(\gamma)\phi = \mathrm{id}$, and writing δ_1 (resp. δ_2) for the T-filler for $T(\beta)f = T(\gamma)\alpha_1$ (resp. $T(\beta)f = T(\gamma)\alpha_2$), there exists unique $\phi' : \delta_1 \to \delta_2$ such that $T(\phi')f = \phi$ and $\gamma\phi' = \mathrm{id}$. With this 2-categorical definition of T-generic morphism at hand, the proof of proposition(2.6) is easily adapted so that it applies to 2-functors. With the naive 2-categorical analogue of p.r.a-ness understood, we now give the more sophisticated analogue which allows for the role of fibrations in our finitely complete 2-categories (see the remarks just before subsection(3.8)).

5.2. Definition. *A 2-functor* $T : \mathcal{A} \to \mathcal{B}$ *between finitely complete 2-categories is* familial *when it is p.r.a and the 2-functor*

$$T_1 : \mathcal{A} \to \mathcal{B}/T1$$

factors through $U_{T1} : \mathrm{Spl}(T1) \to \mathcal{B}/T1$. *T is* opfamilial *when the 2-functor* $T^{\mathrm{co}} : \mathcal{A}^{\mathrm{co}} \to \mathcal{B}^{\mathrm{co}}$ *is familial*[15]. *A 2-monad is familial (resp. opfamilial) when its underlying 2-functor is familial (resp. opfamilial), and its unit and multiplication are cartesian.*

As already discussed in section(3), for a finitely complete 2-category $\mathcal{K}$ and $X \in \mathcal{K}$, one can consider various alternatives to the naive slice $\mathcal{K}/X$ because of the presence of the various concepts of "fibration" internal to $\mathcal{K}$. As in the one dimensional case p.r.a-ness for $T : \mathcal{A} \to \mathcal{B}$ asks more of the assignment

$$A \mapsto Tt_A : TA \to T1,$$

namely, that it is the object map of a right adjoint. Familiality asks further that Tt_A be a split fibration, or more precisely that this right adjoint factor through $\mathrm{Spl}(T1)$. Thus familiality is the 2-categorical analogue of parametric right adjointness provided that one interprets $\mathrm{Spl}(X)$ as the 2-categorical analogue of the slice category of $\mathcal{K}$ over X. In the example of Fam discussed below, Tt_A is one of the most well-known examples of a split fibration: it is the functor

$$\mathrm{Fam}(A) \to \mathrm{Set}$$

which sends a family of objects to its indexing set.

Following the same philosophy one might have considered an alternative to definition(5.2) in which one asked that T_1 factors as

$$\mathcal{A} \xrightarrow{\overline{T}_1} \mathrm{Spl}(T1) \xrightarrow{U_{T1}} \mathcal{B}/T1$$

where $\overline{T}_1$ is a right adjoint. This of course implies that T_1 is a right adjoint since U_{T1} is, but given a mild condition on $\mathcal{A}$ this is in fact a consequence of the original definition.

5.3. Lemma. *If* $T : \mathcal{A} \to \mathcal{B}$ *is a familial 2-functor between finitely complete 2-categories and* $\mathcal{A}$ *has coequalisers, then the 2-functor* $\overline{T}_1$ *has a left adjoint.*

Proof. From the definition one has the following situation

$$\begin{tikzcd}
 & \mathrm{Spl}(T1) \arrow[d, "U_{T1}"] \\
\mathcal{A} \arrow[ur, "\overline{T}_1"] \arrow[r, "T_1"'] \arrow[ur, phantom, "=", shift right=3ex] & \mathcal{B}/T1
\end{tikzcd}$$

to which one may apply Dubuc's adjoint triangle theorem [Dubuc, 1972] to construct the left adjoint to $\overline{T}_1$. ∎

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[15] In other words just reverse all the 2-cells in the definition of "familial" to define "opfamilial". Note that this includes replacing split fibrations by split *op*fibrations.

Another observation which supports the idea that familial 2-functors are a 2-categorical analogue of p.r.a-ness is

5.4. Lemma. *Let $\mathcal{K}$ be a finitely complete 2-category and $B \in \mathcal{K}$. Then the composite*

$$\mathrm{Spl}(B) \xrightarrow{U_B} \mathcal{K}/B \xrightarrow{t_{B!}} \mathcal{K}$$

is familial.

Proof. Note that U_B has a left adjoint F_B and so $t_{B!}U_B$ is p.r.a. Moreover $(t_{B!}U_B)_1$ is just U_B, which trivially factors through U_B and so $t_{B!}U_B$ is familial. ■

Thus denoting the composite $t_{B!}U_B$ by $\bar{t}_B$, one has

5.5. Corollary. *Let $T : \mathcal{A} \to \mathcal{B}$ be a familial 2-functor between finitely complete 2-categories and suppose that $\mathcal{A}$ has coequalisers. Then T factors as a right adjoint followed by a 2-functor of the form $\bar{t}_B$.*

It is sometimes useful to note that the generic morphisms for a familial 2-functor satisfy a stronger 2-dimensional property. We make use of this below in describing Fam's pseudo monad structure and in section(6).

5.6. Definition. *Let $T : \mathcal{A} \to \mathcal{B}$ be a 2-functor between finitely complete 2-categories such that T_1 factors through $U_{T1} : \mathrm{Spl}(T1) \to \mathcal{B}/T1$. Thus for all $A \in \mathcal{A}$, Tt_A is a split fibration. A map $g : B \to TA$ is* lax-T-generic *if for all α, β, γ and ϕ as on the left hand side of the equation below, there are unique δ, ϕ_1 and ϕ_2 so that*

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & TX \arrow[d, "T\gamma"] & & B \arrow[rr, "\alpha"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\phi_1}{\Nearrow}", near start] & & TX \arrow[d, "T\gamma"] \arrow[dl, phantom, "\overset{T\phi_2}{\Nearrow}", near end] \\
TA \arrow[r, "T\beta"'] & TZ \arrow[r, phantom, "="] & {} & TA \arrow[urr, "T\delta"] \arrow[rr, "T\beta"'] & & TZ
\end{tikzcd} \tag{6}$$

and ϕ_1 is chosen Tt_X-cartesian. Reversing the direction of the 2-cells involved gives the definition of oplax-T-generic morphism.

We will now provide some basic useful properties lax-T-genericness, in particular that it implies ordinary T-genericness, and then we will show that the T-generics for familial T are in fact lax.

5.7. Lemma. *Let $T : \mathcal{A} \to \mathcal{B}$ be a 2-functor between finitely complete 2-categories such that T_1 factors through $U_{T1} : \mathrm{Spl}(T1) \to \mathcal{B}/T1$.*

1. *In the situation of equation(6) ϕ is an identity iff ϕ_1 and ϕ_2 are identities.*

2. *If $g : B \to TA$ lax-T-generic then g is T-generic.*

3. *In the situation of equation(6) ϕ is invertible iff ϕ_1 and ϕ_2 are invertible.*

Proof. (1): the implication ($\Leftarrow$) is trivial, so let ϕ be an identity. Now $T(t_X)\phi_1 = T(t_Z)\phi = \mathrm{id}$ and so since ϕ_1 is chosen-$T(t_X)$-cartesian, $\phi_1 = \mathrm{id}$. We have

$$\begin{tikzcd} B \arrow[r, "T(\gamma)\alpha"] \arrow[d, "g"'] \arrow[r, phantom, shift right=2ex, "="] & TZ \arrow[d, "Tt_Z"] \\ TA \arrow[r, "Tt_A"'] \arrow[ur, "T(\gamma\delta)" description] \arrow[r, phantom, shift left=2ex, "="] & T1 \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "T(\gamma)\alpha"] \arrow[d, "g"'] \arrow[r, phantom, shift right=2ex, "="] & TZ \arrow[d, "Tt_Z"] \\ TA \arrow[r, "Tt_A"'] \arrow[ur, "T\beta" description] \arrow[r, phantom, shift left=2ex, "="] & T1 \end{tikzcd}$$

and so by uniqueness $\beta = \gamma\delta$, and applying uniqueness this time to equation(6), one must have $\phi_2 = \mathrm{id}$.
(2): it suffices to verify only the 2-dimensional aspect of genericness as the one-dimensional part is immediate from (1). Given

$$\begin{tikzcd} B \arrow[r, bend left, "\alpha_2"] \arrow[r, bend right, "\alpha_1"'] \arrow[r, phantom, "\phi \Uparrow"] \arrow[d, "g"'] & TX \arrow[d, "T\gamma"] \\ TA \arrow[r, "T\beta"'] & TZ \end{tikzcd}$$

where g satisfies the hypothesis of lemma(5.8), $T(\gamma)\alpha_1 = T(\beta)g = T(\gamma)\alpha_2$, and $T(\gamma)\phi = \mathrm{id}$, we have unique $\delta_1, \delta_2 : A \rightarrow X$ such that

$$\gamma\delta_1 = \beta = \gamma\delta_2 \qquad T(\delta_1)g = \alpha_1 \qquad T(\delta_2)g = \alpha_2$$

We must provide $\phi' : \delta_1 \rightarrow \delta_2$ unique such that $\gamma\phi' = \mathrm{id}$ and $T(\phi')g = \phi$. Since g satisfies the hypothesis of lemma(5.8) we have unique δ_3, ϕ_1 and ϕ_2 so that

$$\phi \quad = \quad \begin{tikzcd} B \arrow[r, "\alpha_2"] \arrow[d, "g"'] \arrow[r, phantom, shift right=3ex, "\phi_1 \Nearrow"] & TX \arrow[d, "T1"] \\ TA \arrow[r, "T\delta_1"'] \arrow[ur, "T\delta_3" description] \arrow[r, phantom, shift left=3ex, "\Nearrow T\phi_2"] & TX \end{tikzcd}$$

and ϕ_1 is chosen-Tt_X-cartesian. Now $T(t_X)\phi_1 = T(t_X)\phi = \mathrm{id}$ and so $\phi_1 = \mathrm{id}$. Composing the above diagram-equation with $T\gamma$ and using uniqueness, one obtains $\gamma\phi_2 = \mathrm{id}$ since $T(\gamma)\alpha = \mathrm{id}$. Thus $\delta_2 = \delta_3$ and $\phi_2 = \phi'$ the required unique 2-cell.
(3): the implication ($\Leftarrow$) is trivial, so let ϕ be an isomorphism. This time ϕ_1 is the cartesian lift of the invertible 2-cell $T(t_Z)\phi$, and so is invertible. Thus $T(\phi_2)g$ is invertible, and so since g is T-generic, ϕ_2 is invertible. ■

5.8. Lemma. *If $T : \mathcal{A} \rightarrow \mathcal{B}$ is familial and $g : B \rightarrow TA$ is T-generic then g is in fact lax-T-generic.*

Proof. Consider α, β, γ and ϕ as on the left hand side of equation(6). Since Tt_X is a split fibration there are $\alpha' : B \rightarrow TX$ and $\phi_1 : \alpha' \rightarrow \alpha$ chosen Tt_X-cartesian such that $T(t_X)\phi_1 = T(t_Z)\phi$. This last equation is an equation of 2-cells, and taking their domain arrows one has $T(t_X)\alpha' = T(t_A)g$, but the genericness of g ensures that $\alpha' = T(\delta)g$ for a unique $\delta : A \to X$. Since ϕ_1 is chosen Tt_X-cartesian and $T\gamma$ is a morphism of split fibrations $Tt_X \to Tt_Z$, $T(\gamma)\phi_1$ is chosen Tt_Z cartesian. Thus there is $\phi'_2 : T(\beta)g \to T(\gamma\delta)g$ unique such that $\phi = (T(\gamma)\phi_1)\phi'_2$, but the 2-dimensional aspect of g's genericness ensures that there is a unique ϕ_2 such that $\phi'_2 = T(\phi_2)g$. ∎

5.9. Fam as a familial 2-functor. For $X \in \mathrm{CAT}$ we recall the explicit definition of the category $\mathrm{Fam}(X)$:

- objects: are pairs (I, f) where $I \in \mathrm{Set}$ and $f : I \to X$ is a functor.
- arrows: an arrow $(I, f) \to (J, g)$ is a pair $(k, \overline{k})$ where $k : I \to J$ is a function and $\overline{k}$ is a natural transformation

$$\begin{tikzcd}
I \arrow[rr, "k"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\overline{k}}{\Rightarrow}"] & J \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

- identities: are those $(k, \overline{k})$ such that k is an identity function and $\overline{k}$ is an identity natural transformation.
- composition: the composite of $(I, f) \xrightarrow{(k,\overline{k})} (J, g) \xrightarrow{(m,\overline{m})} (M, h)$ has function part mk and 2-cell part given by

$$\begin{tikzcd}
I \arrow[r, "k"] \arrow[dr, "f"'] & J \arrow[r, "m"] \arrow[d, "g"] \arrow[d, phantom, shift right=3ex, "\overset{\overline{k}}{\Rightarrow}"] \arrow[d, phantom, shift left=3ex, "\overset{\overline{m}}{\Rightarrow}"] & M \arrow[dl, "h"] \\
& X &
\end{tikzcd}$$

In the obvious way Fam is a 2-functor. A functor $f : I \to X$ is a *family* of objects of X, I is the *indexing set* of this family, and for $i \in I$ we say that $fi \in X$ is the *label* of i. Similarly for $(k, \overline{k})$ as above one may regard the components of $\overline{k}$ as labeling the function k. In more detail, for $i \in I$ the component $\overline{k}_i : fi \to gki$ in X is the *label of the assignment* $i \mapsto ki$. In this way one regards $\mathrm{Fam}(X)$ as the category of *X-labeled sets* and *X-labeled functions.* As already recalled in the previous subsection it is well-known that the functor

$$\mathrm{Fam}(t_X) : \mathrm{Fam}(X) \to \mathrm{Fam}(1){=}\mathrm{Set}$$

given explicitly on objects by $(I, f) \mapsto f$ is a split fibration. The chosen cartesian arrows of the split cleavage for $\mathrm{Fam}(t_X)$ are those $(k, \overline{k})$ such that $\overline{k}$ is an identity. More generally, $(k, \overline{k})$ is $\mathrm{Fam}(t_X)$-cartesian iff $\overline{k}$ is invertible. The cleavage just described is 2-functorial in X, that is one has a factorisation

$$\mathrm{CAT} \longrightarrow \mathrm{Spl(Set)} \xrightarrow{{}^{U}\mathrm{Set}} \mathrm{CAT/Set}$$

of Fam_1 as required by definition(5.2).

In order to establish that Fam is p.r.a we must consider functors $f : B \to \text{Fam}(A)$. By way of notation we shall denote f's object and arrow maps as follows:

$$b \mapsto (fb : Ib \to A) \qquad (\beta : b_1 \to b_2) \mapsto$$

$$\begin{tikzcd}
Ib_1 \arrow[rr, "I\beta"] \arrow[dr, "fb_1"'] & \arrow[d, phantom, "\overset{f\beta}{\Rightarrow}" near start] & Ib_2 \arrow[dl, "fb_2"] \\
& A &
\end{tikzcd}$$

and so I is the functor $\text{Fam}(t_A)f : B \to \text{Set}$. Such a functor f will send each $b \in B$ to an A-labeled set and each morphism of B to an A-labeled function. For any object or arrow of A, one can observe how often it is used as a label in the description of f.

5.10. Definition. *A functor* $f : B \to \text{Fam}(A)$ endows B with elements *when:*

1. *For all* $a \in A$, *there is a unique* $b \in B$ *and a unique* $i \in Ib$ *such that* $fb(i) = a$.
2. *For all* $\alpha : a_1 \to a_2$, *there is a unique* $\beta : b_1 \to b_2$ *in* B *and* $i \in Ib_1$ *such that* $(f\beta)_i = \alpha$.

In other words, f endows B with elements when each object and each arrow of A is used exactly once as a label by f. Thus one may regard $a \in A$ as an element of the unique $b \in B$ such that $fb(i) = a$, and similarly for the arrows of A. We will now see that f is Fam-generic iff it endows B with elements.

5.11. Lemma. *Any* $f : B \to \text{Fam}(A)$ *factors as*

$$B \xrightarrow{g} \text{Fam}(C) \xrightarrow{\text{Fam}(h)} \text{Fam}(B)$$

where g *endows* B *with elements. Moreover if* B *is small and discrete then so is* C.

Proof. We will use the notation for f's object and arrow maps described prior to definition(5.10). Let $C = 1/I$ and define $g : B \to \text{Fam}(C)$ as follows. For $b \in B$, $gb : Ib \to C$ is defined by $gb(i) = i : 1 \to Ib$. For $\beta : b_1 \to b_2$ we write its image by g as

$$\begin{tikzcd}
Ib_1 \arrow[rr, "I\beta"] \arrow[dr, "gb_1"'] & \arrow[d, phantom, "\overset{g\beta}{\Rightarrow}" near start] & Ib_2 \arrow[dl, "gb_2"] \\
& C &
\end{tikzcd}$$

where

$$(g\beta)_i \quad = \quad \begin{tikzcd}
& 1 \arrow[dl, "i"'] \arrow[dr] \arrow[d, phantom, "=" near start] & \\
Ib_1 \arrow[rr, "I\beta"'] & & Ib_2
\end{tikzcd}$$

By definition g endows B with elements. Defining $h : C \to A$ as

$$(i : 1 \to Ib) \mapsto fb(i) \qquad \begin{tikzcd} & 1 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] \arrow[d, phantom, "="] & \\ Ib_1 \arrow[rr, "I\beta"'] & {} & Ib_2 \end{tikzcd} \mapsto \begin{tikzcd} & 1 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] \arrow[d, phantom, "="] & \\ Ib_1 \arrow[rr, "I\beta"] \arrow[dr, "fb_1"'] & {} \arrow[d, phantom, "\overset{f\beta}{\Rightarrow}"] & Ib_2 \arrow[dl, "fb_2"] \\ & A & \end{tikzcd}$$

it follows by definition that $f = \mathrm{Fam}(h)g$. Now C fits into a pullback square

$$\begin{tikzcd} C \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet \arrow[d, "\tau"] \\ B \arrow[r, "I"'] & \mathrm{Set} \end{tikzcd}$$

thus $p : C \to B$ is a discrete opfibration with small fibres, and so if B is small and discrete, then C is small and discrete also. ■

5.12. Lemma. *If $f : B \to \mathrm{Fam}(A)$ endows B with elements then f is* Fam*-generic.*

Proof. Considering

$$\begin{tikzcd} B \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & \mathrm{Fam}(X) \arrow[d, "\mathrm{Fam}(r)"] \\ \mathrm{Fam}(A) \arrow[r, "\mathrm{Fam}(q)"'] & \mathrm{Fam}(Z) \end{tikzcd} \tag{7}$$

we use the notation for f's object and arrow maps described prior to definition(5.10), similarly we use the following notation for p's object and arrow maps,

$$b \mapsto (pb : Jb \to X) \quad (\beta : b_1 \to b_2) \mapsto \begin{tikzcd} Jb_1 \arrow[rr, "J\beta"] \arrow[dr, "pb_1"'] & {} \arrow[d, phantom, "\overset{p\beta}{\Rightarrow}"] & Jb_2 \arrow[dl, "pb_2"] \\ & X & \end{tikzcd}$$

and so J is the functor $\mathrm{Fam}(t_X)p : B \to \mathrm{Set}$. However by the commutativity of (7) we have $Ib = Jb$ and $q(fb) = r(pb)$ for all b, and $I\beta = J\beta$ and $q(f\beta) = r(p\beta)$ for all β. Since f endows B with elements we may define $\delta : A \to X$ as follows. For $a \in A$, let b be the unique object of B and i the unique element of Ib such that $fb(i) = a$, and then $\delta(a) = (pb)(i)$. For $\alpha : a_1 \to a_2$, let $\beta : b_1 \to b_2$ and $i \in Ib_1$ be unique such that $f\beta_i = \alpha$, and then $\delta(\alpha) = (p\beta)_i$. By definition δ is unique such that $\mathrm{Fam}(\delta)f = p$ and $rf = q$. ■

5.13. Proposition. Fam *is a familial 2-functor. Moreover* $f : B \to \mathrm{Fam}(A)$ *is* Fam-*generic iff it endows B with elements.*

Proof. If f endows B with elements then it is generic by lemma(5.12). Thus by lemma(5.11) Fam is p.r.a. Moreover by that lemma if f is generic, then you can factor it as $\mathrm{Fam}(h)g$ where g endows B with elements. But this implies that g is itself generic, and so h is an isomorphism, whence f also endows B with elements. ∎

5.14. Corollary. Fam *is a polynomial 2-functor:* $\mathrm{Fam} \cong \mathcal{P}_\tau$*, where* $\tau : \mathrm{Set}_\bullet \to \mathrm{Set}$ *is the forgetful functor from the category of pointed sets.*

Proof. Since Fam is p.r.a it suffices to show that L_{Fam} is isomorphic to the composite

$$\mathrm{CAT}/\mathrm{Set} \xrightarrow{\tau^*} \mathrm{CAT}/\mathrm{Set} \xrightarrow{(t_{\mathrm{Set}})_!} \mathrm{CAT} .$$

By proposition(5.13) the generic factorisation of $f : B \to \mathrm{Fam}(A)$ was being constructed in lemma(5.11). Applying this construction in the case $A = 1$, gives the result by proposition(2.6). ∎

To see that familiality is stronger than p.r.a'ness, note that by theorem(6.2) below, familial 2-functors preserve discrete fibrations, and dually, opfamilial 2-functors preserve discrete opfibrations. However applying Fam to the discrete opfibration $\tau : \mathrm{Set}_\bullet \to \mathrm{Set}$ does not give a discrete opfibration. Consider for instance a set S with 2 distinct elements x and y. Then the 2-element family of pointed sets $((x, S), (y, S))$ is sent by $\mathrm{Fam}(\tau)$ to the 2-element family (S, S). There is a unique chosen $\mathrm{Fam}(t_{\mathrm{Set}})$-cartesian map $f : (S, S) \to (S)$ in Fam(Set), where (S) denotes the singleton family consisting of the set S, and this map admits no lifting to a map $((x, S), (y, S)) \to ((z, S))$ in $\mathrm{Fam}(\mathrm{Set}_\bullet)$: if such a lifting existed we would have $x = z = y$, but x and y are different. Thus Fam is familial but not opfamilial, and dually the endo-2-functor of CAT whose object map is $X \mapsto \mathrm{Fam}(X^{\mathrm{op}})^{\mathrm{op}}$ is opfamilial but not familial. An example of a p.r.a 2-functor which is neither familial nor opfamilial is Φ_B, the underlying endofunctor of the fibrations 2-monad of subsection(3.1). Even in the case $\mathcal{K} = \mathrm{CAT}$ one can easily verify that Φ_B is neither familial nor opfamilial.

It is well known that Fam is the underlying 2-functor of a lax-idempotent pseudo-monad. The only thing that is pseudo about the monad we describe below, is that the associative law of the monad holds up to a canonical isomorphism. The other monad laws can be made to hold strictly. However when one considers a finite analogue of Fam, which we denote Fam_f, we do indeed obtain an honest example of a familial 2-monad. We shall now recover Fam's pseudo monad structure, and also show that the unit and multiplication of this monad structure is cartesian. Interestingly we are able to obtain these facts from our understanding of Fam-generics.

The unit $\eta_X : X \to \mathrm{Fam}(X)$ picks out singleton families, that is,

$$x \mapsto (x : 1 \to X) \qquad (f : x \to x') \mapsto \begin{tikzcd} 1 \arrow[rr, "1"] \arrow[dr, "x"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & 1 \arrow[dl, "x'"] \\ & X & \end{tikzcd}$$

and η so defined is clearly a cartesian transformation. Now we shall fix a choice of Fam-generic factorisation and use the notation of scholium(2.7). Without loss of generality we assume that

1. the components of η are chosen generics.
2. if $f : I \to X$ in $\mathrm{Fam}(X)$, then

$$\begin{tikzcd} 1 \arrow[r, "1_I"] & \mathrm{Fam}(I) \arrow[r, "\mathrm{Fam}(f)"] & \mathrm{Fam}(X) \end{tikzcd}$$

   is the chosen generic factorisation of $f : 1 \to \mathrm{Fam}(X)$.

An object of $\mathrm{Fam}^2(X)$ consists of $I \in$ Set and a functor $f : I \to \mathrm{Fam}(X)$. We have our chosen generic factorisation

$$\begin{tikzcd} I \arrow[r, "g_f"] & \mathrm{Fam}(D_f) \arrow[r, "\mathrm{Fam}(h_f)"] & \mathrm{Fam}(X) \end{tikzcd}$$

of f, and $D_f \in$ Set by lemma(5.11). We define $\mu_X(f) = h_f$. The arrow map of μ_X is given by

$$\begin{tikzcd} I_1 \arrow[rr, "\alpha"] \arrow[dr, "f_1"'] & \arrow[d, phantom, "\overset{\overline{\alpha}}{\Rightarrow}"] & I_2 \arrow[dl, "f_2"] & & D_{f_1} \arrow[rr, "\delta"] \arrow[dr, "h_{f_1}"'] & \arrow[d, phantom, "\overset{\alpha_2}{\Rightarrow}"] & D_{f_2} \arrow[dl, "h_{f_2}"] \\ & \mathrm{Fam}(X) & & \mapsto & & X & \end{tikzcd}$$

where

$$\overline{\alpha} \quad = \quad \begin{tikzcd} I_1 \arrow[rr, "g_{f_2}\alpha"] \arrow[d, "g_{f_1}"'] & \arrow[d, phantom, "\overset{\alpha_1}{\Nearrow}"] & \mathrm{Fam}(D_{f_2}) \arrow[d, "\mathrm{Fam}(h_{f_2})"] \\ \mathrm{Fam}(D_{f_2}) \arrow[urr, "\mathrm{Fam}(\delta)" description] \arrow[rr, "\mathrm{Fam}(h_{f_1})"'] & \arrow[ur, phantom, "\overset{\mathrm{Fam}(\alpha_2)}{\Nearrow}"] & \mathrm{Fam}(X) \end{tikzcd}$$

where α_1 is chosen-$\mathrm{Fam}(t_{D_{f_2}})$-cartesian.

5.15. PROPOSITION. $(\mathrm{Fam}, \eta, \mu)$ *is a lax idempotent pseudo-monad, whose underlying 2-functor is familial and whose unit and multiplication are cartesian.*

PROOF. The verification that μ is a cartesian 2-natural transformation follows easily from the definition of μ and lemma(5.8). The unit laws

$$\mu_X \mathrm{Fam}(\eta_X) = \mathrm{id} = \mu_X \eta_{\mathrm{Fam}(X)}$$

follow from the assumptions we imposed on our chosen Fam-generic factorisations. To finish the proof we provide $c : 1 \to \eta_{\mathrm{Fam}} \mu$ such that $\mu c = \mathrm{id}$ and $c\eta_{\mathrm{Fam}} = \mathrm{id}$, because then c is the unit for an adjunction $\mu \dashv \eta_{\mathrm{Fam}}$ with identity counit. For $X \in$ CAT consider $f : I \to \mathrm{Fam}(X)$ in $\mathrm{Fam}^2(X)$. We have $h_f : D_f \to X$ with $D_f \in$ Set, and then

$\eta_{\mathrm{Fam}(X)}\mu_X(I,f) = h_f : 1 \to \mathrm{Fam}(X)$. By the description of Fam-generic factorisations given in lemma(5.11) , we have for $i \in I$

$$\begin{tikzcd}
I(fi) \arrow[rr, "c_i"] \arrow[dr, "fi"'] & {} \arrow[d, phantom, "="] & D_f \arrow[dl, "h_f"] \\
& X &
\end{tikzcd}$$

where the c_i form a coproduct cocone. Regarding the above triangle as a morphism of $\mathrm{Fam}(X)$, in fact a chosen $\mathrm{Fam}(t_X)$-cartesian morphism since the 2-cell part is an identity, these morphisms provide the components of

$$\begin{tikzcd}
I \arrow[rr, "t_I"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\overset{\overline{c}_f}{\Longrightarrow}"] & 1 \arrow[dl, "h_f"] \\
& \mathrm{Fam}(X) &
\end{tikzcd}$$

which we regard as an arrow of $\mathrm{Fam}^2(X)$. These are the components of $c : 1 \to \eta_{\mathrm{Fam}}\mu$. To see that $\mu_X c_X = \mathrm{id}$, note that $\overline{c}$ is chosen cartesian, and so by the definition of μ, will be sent by μ_X to identities. To see that $c_X\eta_X = \mathrm{id}$ one must show that $\overline{c}_f$ is an identity whenever $I = 1$, but this also follows from the assumptions we imposed on our chosen Fam-generic factorisations. Note that the isomorphism $\mu\mathrm{Fam}(\mu) \cong \mu\mu_{\mathrm{Fam}}$ is obtained by adjointness from $\eta_{\mathrm{Fam}}\eta = \mathrm{Fam}(\eta)\eta$ which holds by the naturality of η. ■

5.16. Example. We consider now a variation on Fam, where $\mathrm{Fam}_f(X)$ is defined as Fam is, except that the indexing sets are only allowed to be natural numbers n, regarded as sets:

$$n = \{0, 1, ..., n-1\}.$$

All of the above analysis of Fam proceeds in the same way for Fam_f, except that $\mathrm{Fam}_f(1)$ is the category of natural numbers and functions between them (instead of Set), and the isomorphism $\mu\mathrm{Fam}(\mu) \cong \mu\mu_{\mathrm{Fam}}$ can be strictified. The reason for this is that coproducts in $\mathrm{Fam}(1) = \mathrm{Set}$, which are coherently associative and unital, are replaced by the strictly associative and unital ordinal sums in $\mathrm{Fam}_f(1)$. Thus $(\mathrm{Fam}_f, \eta, \mu)$ is a lax-idempotent familial 2-monad.

6. Basic properties of familial 2-functors

In this section we establish the basic properties that familial 2-functors satisfy. In particular we study the senses in which these 2-functors preserve lax and pseudo pullbacks. Moreover we will see how familial 2-functors preserve all the different notions of fibration in a finitely complete 2-category which have been discussed so far.

As a parametric right 2-adjoint any familial 2-functor must preserve pullbacks. We shall now discuss how familial 2-functors are also well-behaved with respect to lax and pseudo pullbacks. Recall that a *right adjoint retraction* for an arrow $f : A \rightarrow B$ in a 2-category $\mathcal{K}$ is an adjunction $f \dashv r$ in $\mathcal{K}$ such that the unit is an identity. Such a retraction for f is determined by the data: $r : B \rightarrow A$ and $\varepsilon : fr \rightarrow 1_B$, satisfying the equations:

$$rf = 1_A \qquad r\varepsilon = \mathrm{id} \qquad \varepsilon f = \mathrm{id}$$

When ε is invertible r is a *pseudo-inverse retraction* for f, and in this case f and r are part of an adjoint equivalence. Similarly a left adjoint retraction for f is an adjunction $l \dashv f$ whose counit is an identity. In [Weber, 2007] a 2-functor $T : \mathcal{A} \rightarrow \mathcal{B}$ between finitely complete 2-categories was said to *preserve lax pullbacks up to a left adjoint section* when for all

$$\begin{tikzcd} A \arrow[r, "f"] & C & B \arrow[l, "h"'] \end{tikzcd}$$

in $\mathcal{A}$, the comparison map $k : T(f/h) \rightarrow Tf/Th$ has a right adjoint retraction in $\mathcal{B}/TA$. It is also interesting to consider the case when k has a pseudo-inverse in $\mathcal{B}/TA$, in which case T is said to *preserve lax pullbacks up to a left equivalence section.* Note that this is different to asking that T preserve lax pullbacks up to a *right* equivalence section because the relevant equivalence lives in a different slice. One can of course make all of the analogous definitions for pseudo pullbacks as well, and we shall freely use the corresponding terminology.

6.1. Theorem. *If $T : \mathcal{A} \rightarrow \mathcal{B}$ is a familial 2-functor between finitely complete 2-categories.*

1. *T preserves lax pullbacks up to a left adjoint section.*
2. *T preserves pseudo pullbacks up to a left equivalence section.*

Proof. (1): For $f : A \rightarrow C$ and $h : B \rightarrow C$ in $\mathcal{A}$ we have lax pullbacks

$$\begin{tikzcd} f/h \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[d, "h"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \qquad \qquad \begin{tikzcd} Tf/Th \arrow[r, "q_2"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{\lambda_2}{\Rightarrow}"] & TB \arrow[d, "Th"] \\ TA \arrow[r, "Tf"'] & TC \end{tikzcd}$$

and the comparison $k : T(f/h) \rightarrow Tf/Th$ is the unique map such that $\lambda_2 k = T\lambda$. Generically factorising p_2 as $p_2 = T(s)g$ and by lemma(5.8) one obtains

$$\lambda_2 \quad = \quad \begin{tikzcd} Tf/Th \arrow[rr, "q_2"] \arrow[dd, "g"'] & & TB \arrow[dddd, "Th"] \\ & \overset{\phi_1}{\Nearrow} & \\ TD \arrow[uurr, bend right, "T\delta"'] \arrow[dd, "Ts"'] & & \\ & \overset{T\phi_2}{\Nearrow} & \\ TA \arrow[rr, "Tf"'] & & TB \end{tikzcd}$$

where ϕ_1 is chosen-$T(t_C)$-cartesian. Letting $t : D \to f/h$ be the unique map such that $\lambda t = \phi_2$, we define $r : Tf/Th \to T(f/h)$ as $r = T(t)g$. We have the 2-cells $\mathrm{id} : p_2kr \to p_2$ and $\phi_1 : q_2kr \to q_2$, and note that

$$\begin{tikzcd}
T(f)p_2kr \arrow[r, "\mathrm{id}"] \arrow[d, "\lambda_2kr"'] & T(f)p_2 \arrow[d, "\lambda_2"] \\
T(h)q_2kr \arrow[r, "T(h)\phi_1"'] & T(h)q_2
\end{tikzcd}$$

commutes since $\lambda_2kr = T(\lambda tg = T(\phi_2)g$ and by the above factorisation of λ_2. Thus we can define $\varepsilon : kr \to 1$ as the unique 2-cell such that $p_2\varepsilon = \mathrm{id}$ and $q_2\varepsilon = \phi_1$. By definition r and ε can be regarded as living in $\mathcal{B}/TA$, so it suffices to verify: $rk = 1$, $\varepsilon k = \mathrm{id}$ and $r\varepsilon = \mathrm{id}$.

The key observation which enables us to perform these verifications is that since T is a parametric right 2-adjoint, the 2-cell $T(\lambda)$ is a lax pullback in $\mathcal{B}/T1$, because it is the result of applying the right 2-adjoint T_1 to the lax pullback λ.

To see that $rk = 1$ first note that we have

$$T(\lambda)rk = T(\lambda t)gk = T(\phi_2)gk \qquad T(\lambda) = (T(\lambda)rk)(\phi_1 k)$$

Now $\phi_1 k$ is chosen-$T(t_C)$-cartesian and $T(t_C)\phi_1 = T(t_B)T(\lambda) = \mathrm{id}$, whence $\phi_1 k = \mathrm{id}$. Thus we have $T(\lambda) = T(\lambda)rk$. Now rk may be regarded as an arrow of $\mathcal{B}/T1$ because $T(t_{f/h})rk = T(t_D)gk = T(t_Ap) = T(t_{f/h})$, and so we may exploit the universal property of $T(\lambda)$ as a lax pullback in $\mathcal{B}/T1$ to infer that $rk = 1$.

By the definition of ε, εk is the unique 2-cell such that $p_2\varepsilon k = \mathrm{id}$ and $q_2\varepsilon k = \phi_1 k = \mathrm{id}$. Thus $\varepsilon k = \mathrm{id}$. To see that $r\varepsilon = \mathrm{id}$ notice that

$$\begin{tikzcd}
Tf/Th \arrow[dr, bend left, "r"] \arrow[dr, bend right, "rkr"'] \arrow[dr, phantom, "r\varepsilon \Nearrow"] & & & & \\
& T(f/h) \arrow[r, "Tq"] \arrow[d, "Tp"'] \arrow[dr, phantom, "\overset{T\lambda}{\Longrightarrow}"] & TB \arrow[d, "Th"] & = & T(\lambda)r \\
& TA \arrow[r, "Tf"'] & TC & &
\end{tikzcd}$$

since $T(p)r\varepsilon = p_2\varepsilon = \mathrm{id}$. Regarding the object Tf/Th as over $T1$ via the map $T(t_d)g : Tf/Th \to T1$, r is in $\mathcal{B}/T1$ since $T(t_{f/h})r = T(t_D)g$, and rkr and $r\varepsilon$ are also in $\mathcal{B}/T1$ since $T(t_{f/h}r\varepsilon = T(t_D)g\varepsilon = T(t_A)p_2\varepsilon = \mathrm{id}$. Thus one may exploit the universal property of $T(\lambda)$ as a lax pullback in $\mathcal{B}/T1$ to infer from the previous diagram that $r\varepsilon = \mathrm{id}$.
(2): by lemma(5.7)(3) when one replaces lax by pseudo pullbacks in the proof of (1), all the induced 2-cells will be invertible, and so one obtains a proof of this result. ■

We shall explain how familial 2-functors preserve the various notions of fibration that one can consider. Since most of these notions have an adjoint characterisation, we can use the previous result to help establish this.

6.2. THEOREM. *Let $T : \mathcal{A} \to \mathcal{B}$ be a familial 2-functor between finitely complete 2-categories. Then T preserves fibrations, bifibrations, iso-fibrations and one-sided discrete fibrations.*

PROOF. Let $f : E \to A$ be in $\mathcal{A}$. If f is a fibration then the map $\eta_f : E \to A/f$ has a right adjoint a in $\mathcal{A}/A$. Since $\eta_{Tf} = kT(\eta_f)$ and

$$\begin{tikzcd}
TE \arrow[r, shift left, "T\eta_f"] \arrow[r, phantom, "\perp"] & T(A/f) \arrow[l, shift left, "Ta"] \arrow[r, shift left, "k"] \arrow[r, phantom, "\perp"] & TA/Tf \arrow[l, shift left, "r"]
\end{tikzcd}$$

are adjoints in $\mathcal{B}/TA$, where k is the comparison map, the composite adjunction exhibits Tf as a fibration. The case of iso-fibrations is identical except for the replacement of lax pullbacks by pseudo pullbacks. If f is a bifibration then the canonical map $i_1 : A/_{\cong}f \to A/f$ has a right adjoint a in $\mathcal{A}/A$. Thus we have

$$\begin{tikzcd}
T(A/_{\cong}f) \arrow[r, shift left, "Ti_1"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "k_1"'] & T(A/f) \arrow[l, shift left, dotted, "Ta"] \arrow[d, shift right, "k_2"'] \arrow[d, phantom, shift left, "\dashv"] \\
TA/_{\cong}Tf) \arrow[r, "i_2"'] \arrow[u, shift right, dotted, "r_1"'] & T(A/f) \arrow[u, shift right, dotted, "r_2"']
\end{tikzcd}$$

in $\mathcal{B}/TA$, where the k_j are comparison maps, r_1 is a pseudo inverse for k_1 and the square involving just the solid arrows commutes. Thus $i_2 \dashv k_1T(a)r_2$ exhibits Tf as a bifibration. Now suppose that f is a discrete fibration and consider

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\overset{\phi}{\Longrightarrow}"] & TE \arrow[dl, "Tf"] \\
& TA &
\end{tikzcd}$$

in $\mathcal{B}$. Fix a generic factorisation $\beta = gT(h)$ and by lemma(5.8) we obtain

$$\phi \quad = \quad \begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\phi_1 \Nearrow"] & TE \arrow[d, "T\gamma"] \\
TC \arrow[r, "Th"'] \arrow[ur, "T\delta"] \arrow[ur, phantom, shift right=3, "\Nearrow T\phi_2"] & TA
\end{tikzcd}$$

for unique ϕ_1, ϕ_2 and δ with ϕ_1 chosen-Tt_A-cartesian. Since f is a discrete fibration we have $\phi_3 : \delta_2 \to \delta$ unique such that $f\phi_3 = \phi_2$. Thus we have $\overline{\phi} = (T(\phi_3)g)(\phi_1)$ with $T(f)\overline{\phi} = \phi$. To see that $\overline{\phi}$ is a unique lifting suppose that we have

$$\phi \quad = \quad \begin{tikzcd}
X \arrow[rr, bend left, "\alpha"] \arrow[rr, phantom, bend left=15, "\phi_4 \Nearrow"] \arrow[rr, bend right=15, "\gamma"'] \arrow[dr, "\beta"'] & & TE \arrow[dl, "Tf"] \\
& TA \arrow[u, phantom, "="] &
\end{tikzcd}$$

Then $\gamma = T(\delta_3)g$ for a unique δ_3 such that $h = f\delta_3$, and we obtain ϕ_5, δ_4 and ϕ_6 unique such that

$$\phi_4 \quad = \quad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "g"'] \arrow[r, phantom, shift right=4, "\phi_5\Uparrow" near start] & TE \arrow[d, "T1"] \\ TC \arrow[ur, "T\delta_4" description] \arrow[r, "T\delta_3"'] \arrow[r, phantom, shift left=4, "\Uparrow T\phi_6" near end] & TE \end{tikzcd}$$

ϕ_5 is chosen-Tt_A-cartesian. Composing this last equation with Tf and applying uniqueness we obtain $\phi_5 = \phi_1$ and $f\phi_6 = \phi_2$. Since f is a discrete fibration this last equation implies that $\phi_6 = \phi_3$ whence $\phi_4 = \overline{\phi}$. ■

To see that a familial 2-functor $T : \mathcal{A} \to \mathcal{B}$ also preserves split fibrations, we look more closely at how a cleavage for Tf was constructed from a cleavage for f. In the following lemma we shall characterise the chosen-Tf-cartesian maps in terms of the chosen-f-cartesian maps. To this end we let $f : E \to A$ be a fibration in $\mathcal{A}$ with cleavage (a, ε) and

$$\begin{tikzcd} X \arrow[r, bend left, "\alpha_1"{name=U}] \arrow[r, bend right, "\alpha_2"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\phi"] & TE \end{tikzcd}$$

be in $\mathcal{B}$. Take generic factorisations $\alpha_1 = T(h_1)g_1$ and $\alpha_2 = T(h_2)g_2$ so that we have

$$\phi \quad = \quad \begin{tikzcd} X \arrow[r, "g_2"] \arrow[d, "g_1"'] \arrow[r, phantom, shift right=4, "\phi_1\Uparrow" near start] & TD_2 \arrow[d, "Th_2"] \\ TD_1 \arrow[ur, "T\delta" description] \arrow[r, "Th_1"'] \arrow[r, phantom, shift left=4, "\Uparrow T\phi_2" near end] & TE \end{tikzcd}$$

for unique δ, ϕ_1 and ϕ_2 with ϕ_1 chosen-T-cartesian.

6.3. Lemma. *In the situation just described ϕ is chosen-Tf-cartesian iff ϕ_2 is chosen-f-cartesian.*

Proof. We have lax pullbacks

$$\begin{tikzcd} A/f \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & E \arrow[d, "f"] \\ A \arrow[r, "1"'] & A \end{tikzcd} \qquad\qquad \begin{tikzcd} TA/Tf \arrow[r, "q_2"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{\lambda_2}{\Rightarrow}"] & TE \arrow[d, "Tf"] \\ TA \arrow[r, "1"'] & TA \end{tikzcd}$$

and recall that $q\varepsilon = \lambda$ and that

$$\begin{tikzcd} Z \arrow[r, bend left, "\kappa_1"{name=U}] \arrow[r, bend right, "\kappa_2"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\psi"] & TE \end{tikzcd}$$

is chosen-f-cartesian iff $\psi = q\varepsilon\psi'$, where $\psi' : Z \rightarrow A/f$ is defined by $\lambda\psi' = f\psi$. As in the proof of theorem(6.1) one can express λ_2 as the composite

$$\begin{tikzcd}
TA/Tf \arrow[rr, "q_2"] \arrow[d, "g"'] \arrow[dr, phantom, "\lambda_3\Uparrow"] & & TE \arrow[ddd, "Tf"] \\
TC \arrow[urr, "T\delta_2" description] \arrow[dr, "Ts" description] \arrow[dd, "Th"'] \arrow[r, phantom, "="] & \arrow[r, phantom, "T(q\varepsilon)\Nwarrow"] & {} \\
{} \arrow[r, phantom, "="] & T(A/f) \arrow[uur, bend left, "Tq" description] \arrow[uur, bend right, "Ta" description] \arrow[dl, "Tp"] \arrow[dr, phantom, "="] & {} \\
TA \arrow[rr, "1"'] & & TA
\end{tikzcd}$$

where $p_2 = T(h)g$ is a generic factorisation and r, the right adjoint to the comparison map $k : T(A/f) \rightarrow TA/Tf$ is defined as $r = T(s)g$. The counit ε_2 of $k \dashv r$ is defined by $p_2\varepsilon_2 = \text{id}$ and $q_2\varepsilon_2 = \lambda_3$. From the proof of theorem(6.2) the induced cleavage is given by $(\overline{a}, \overline{\varepsilon})$ where

$$\overline{a} = T(as)g \qquad \overline{\varepsilon} = \varepsilon_2(kT(\varepsilon s)g)$$

Notice how the previous diagram expresses $\lambda_2 = q_2\overline{\varepsilon}$. Define $\phi' : X \rightarrow TA/Tf$ as the unique map such that $p_2\phi' = T(f)\alpha_1$, $q_2\phi' = \alpha_2$ and $\lambda_2\phi' = T(f)\phi$. Define δ_3 as in

$$\begin{tikzcd}
X \arrow[r, "\phi'"] \arrow[d, "g_1"'] & TA/Tf \arrow[d, "g"] \\
TD_1 \arrow[r, dotted, "T\delta_3" description] \arrow[d, "Th_1"'] & TC \arrow[d, "Th"] \\
TE \arrow[r, "Tf"'] & TA
\end{tikzcd}$$

such that $g\phi' = T(\delta_3)g_1$ and $h\delta_3 = fh_1$. The pasting composite of the previous two diagrams is $T(f)\phi$, but $T(f)\phi$ is also the composite of

$$\begin{tikzcd}
X \arrow[r, "g_2"] \arrow[d, "g_1"'] & TD_2 \arrow[r, "Th_2"] \arrow[d, "Th_2" description] \arrow[r, phantom, shift right=6, "="] & TE \arrow[d, "Tf"] \\
TD_1 \arrow[ur, "T\delta" description] \arrow[r, "Th_1"'] \arrow[u, phantom, shift left=6, "\phi_1\Nearrow"] & TE \arrow[r, "Tf"'] \arrow[ul, phantom, shift left=4, "T\phi_2\Nearrow"] & TA
\end{tikzcd}$$

and so by the uniqueness part of lemma(5.8) we have

$$h_2\delta = \delta_2\delta_3 \qquad T(h_2)\phi_1 = \lambda_3\phi' \qquad f\phi_2 = fq\varepsilon s\delta_3$$

Now ϕ is chosen-Tf-cartesian iff $q_2\overline{\varepsilon}\phi' {=} \phi$ which is true iff

$$h_2\delta = \delta_2\delta_3 \qquad T(h_2)\phi_1 = \lambda_3\phi' \qquad \phi_2 = q\varepsilon s\delta_3$$

by the uniqueness part of lemma(5.8). Since the first two equations hold automatically, this is equivalent to saying that $\phi_2 = q\varepsilon s\delta_3$, which exhibits ϕ_2 as chosen-f-cartesian. Conversely if ϕ_2 is chosen-f-cartesian then there is a map $t : D{\rightarrow}A/f$ such that $q\varepsilon t = \phi_2$, so that $\lambda s\delta_3 = f\phi_2 = \lambda t$, whence $s\delta_3 = t$ by the universal property of λ, and so $\phi_2 = q\varepsilon s\delta_3$, which we saw above is equivalent to saying that ϕ is chosen-Tf-cartesian. ■

6.4. Remark. The characterisation of chosen-Tf-cartesians given in lemma(6.3) is independent of the generic factorisations of α_1 and α_2 chosen at the outset.

Proof. Choose alternative generic factorisations $\alpha_1 = T(h'_1)g'_1$ and $\alpha'_2 = T(h'_2)g'_2$, and then form δ', ϕ'_1 and ϕ'_2 in the same way. Thus ϕ is the following composite

$$\begin{tikzcd}
& & X \arrow[dll, bend right, "g_1"'] \arrow[dl, "g'_1"'] \arrow[dr, "g'_2"] \arrow[drr, bend left, "g_2"] & & \\
TD_1 \arrow[r, dotted, "T\iota_1"] \arrow[drr, bend right, "Th_1"'] \arrow[urr, phantom, "="] \arrow[drr, phantom, "="'] & TD'_1 \arrow[rr, "T\delta'"] \arrow[dr, "Th'_1"'] \arrow[rr, phantom, shift left=5ex, "\overset{\Rightarrow}{\phi'_1}"] \arrow[rr, phantom, shift right=5ex, "\overset{T\phi'_2}{\Rightarrow}"] & & TD'_2 \arrow[r, dotted, "T\iota_2"] \arrow[dl, "Th'_2"] & TD_2 \arrow[dll, bend left, "Th_2"] \arrow[ull, phantom, "="'] \arrow[dll, phantom, "="] \\
& & TE & &
\end{tikzcd}$$

where ι_1 and ι_2 were induced uniquely so that the commutative regions of the above diagram are obtained. Since all the maps g_1, g'_1, g_2 and g'_2 are T-generic, ι_1 and ι_2 are in fact invertible. Since $\phi'_1\iota_1$ is chosen-Tt_{D_2}-cartesian, this factorisation of ϕ coincides with the original one described prior to lemma(6.3). By the uniqueness part of lemma(5.8), one obtains $\phi_2 = \phi'_2\iota_1$. Since ι_1 is invertible this last equation implies that ϕ_2 is chosen-f-cartesian iff ϕ'_2 is. ■

6.5. Corollary. *If $T : \mathcal{A}{\rightarrow}\mathcal{B}$ is a familial 2-functor between finitely complete 2-categories then T preserves split fibrations.*

Proof. It suffices to show that the assignment $(a, \varepsilon) \mapsto (\overline{a}, \overline{\varepsilon})$ described in lemma(6.3) sends split cleavages to split cleavages, so we suppose that (a, ε) is a split cleavage for $f : E{\rightarrow}A$. If $\phi : \alpha_1{\rightarrow}\alpha_2$ as above is an identity, then ϕ_2 is an identity by lemma(5.7), which is chosen-f-cartesian since (a, ε) is split, and so by lemma(6.3) ϕ is chosen-Tf-cartesian. Suppose that $\phi : \alpha_1{\rightarrow}\alpha_2$ as above and $\psi : \alpha_2{\rightarrow}\alpha_3$ are chosen-Tf-cartesian. Factoring ψ as we did ϕ we have

$$\psi\phi \quad = \quad \begin{tikzcd}
& X \arrow[dl, "g_1"'] \arrow[d, "g_2"] \arrow[dr, "g_3"] & \\
TD_1 \arrow[r, "T\delta"] \arrow[dr, "Th_1"'] \arrow[r, phantom, shift left=5ex, "\overset{\Rightarrow}{\phi_1}"] \arrow[r, phantom, shift right=5ex, "\overset{T\phi_2}{\Rightarrow}"] & TD_2 \arrow[r, "T\delta'"] \arrow[d, "Th_2"] \arrow[r, phantom, shift left=5ex, "\overset{\Rightarrow}{\psi_1}"] \arrow[r, phantom, shift right=5ex, "\overset{T\psi_2}{\Rightarrow}"] & TD_3 \arrow[dl, "Th_3"] \\
& TE &
\end{tikzcd}$$

where ϕ_2 and ψ_2 are chosen-f-cartesian by lemma(6.3). Now $(\psi_2\delta)\phi_2$ is chosen-f-cartesian since (a, ε) is split, and so $\psi\phi$ is chosen-Tf-cartesian by lemma(6.3). ■

The preservation of split fibrations by familial 2-functors is expressed more comprehensively in the following result.

6.6. Theorem. *If $T : \mathcal{A} \rightarrow \mathcal{B}$ is a familial 2-functor between finitely complete 2-categories then for all $A \in \mathcal{A}$, the effect of T on morphisms*

$$T_A : \mathcal{A}/A \rightarrow \mathcal{B}/TA$$

can be lifted to 2-functors between 2-categories of split fibrations as in:

$$\begin{tikzcd}
\mathrm{Spl}(A) \arrow[r, dotted] \arrow[d, "U_A"'] \arrow[dr, phantom, "="] & \mathrm{Spl}(TA) \arrow[d, "U_{TA}"] \\
\mathcal{A}/A \arrow[r, "T_A"'] & \mathcal{B}/TA
\end{tikzcd}$$

Proof. The proofs of lemma(6.3) and corollary(6.5) provide the object mapping of the desired lifting, and since all 2-cells between morphisms of split fibrations are 2-cells of split fibrations, it suffices to provide the 1-cell mapping to finish defining the lifted 2-functor. Let $f : E \rightarrow A$ be a split fibration and ϕ be given as above. Suppose also that $f' : C \rightarrow A$ is another split fibration with a given cleavage, and that $k : f \rightarrow f'$ is a morphism of $\mathrm{Spl}(A)$. If ϕ is chosen-Tf-cartesian, then ϕ_2 is chosen-f-cartesian and so $k\phi_2$ is chosen-f'-cartesian since k is a morphism of split fibrations. But $k\phi_2 = (T(k)\phi)_2$ by the uniqueness part of lemma(5.8), and so $T(k)\phi$ is chosen-$T(f')$-cartesian by lemma(6.3). That is, Tk is indeed a map of split fibrations. ■

This lifting will be denoted as $T_A : \mathrm{Spl}(A) \rightarrow \mathrm{Spl}(TA)$.

7. Examples of familial 2-functors

So far we have exhibited a few fundamental examples of familial 2-functors. We shall now discuss results which enable us to exhibit more examples. In particular we exhibit the underlying endofunctors of the 2-monads which describe higher symmetric and braided operads within the framework of [Weber, 2005] as being familial and opfamilial.

7.1. Proposition. *Suppose that the 2-categories $\mathcal{A}$ and $\mathcal{B}$ are finitely complete. If $T : \mathcal{A} \rightarrow \mathcal{B}$ is a parametric right 2-adjoint and T preserves lax pullbacks then T is familial and opfamilial.*

Proof. It suffices to show that T is familial: reversing the 2-cells will give opfamiliality. First note that $A \in \mathcal{A}$ is discrete iff

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & A \arrow[d, "1"] \\
A \arrow[r, "1"'] & A
\end{tikzcd}$$

is a lax pullback, so T preserves discrete objects, and thus $T1$ is discrete. Thus for all $A \in \mathcal{A}$, $T(t_A)$ is a split fibration for which the chosen cartesian cells are identities, and for all $f : A_1 \to A_2$, Tf obviously preserves these chosen cartesian cells, and so T_1 factors through $U_{T1} : \mathrm{Spl}(T1) \to \mathcal{B}/T1$ as required. ■

7.2. Corollary. *Let* $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ *be a parametric right adjoint. Then*

$$\mathrm{Cat}(T) : \mathrm{CAT}(\widehat{\mathbb{C}}) \to \mathrm{CAT}(\widehat{\mathbb{D}})$$

is familial and opfamilial.

Proof. Since T preserves pullbacks, and lax pullbacks in $\mathrm{CAT}(\widehat{\mathbb{C}})$ and $\mathrm{CAT}(\widehat{\mathbb{D}})$ can be constructed in terms of pullbacks in $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{D}}$ respectively, $\mathrm{Cat}(T)$ preserves lax pullbacks. By the previous proposition, it suffices to show that $\mathrm{Cat}(T)$ is a parametric right 2-adjoint. This follows by applying $\mathrm{Cat}(-)$ to the factorisation

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "T_1"] & \widehat{\mathbb{D}}/T1 \arrow[r, "t_{T1}"] & \widehat{\mathbb{D}}
\end{tikzcd}$$

because $\mathrm{Cat}(-)$ applied to t_{T1} is the equivalence $\mathrm{Cat}(\widehat{\mathbb{D}}/T1) \simeq \mathrm{Spl}(T1)$ followed by $\bar{t}_{T1}$ which is p.r.a by lemma(5.4). ■

7.3. Examples. An ω-operad in Span the sense of [Batanin, 1998] gives rise to a p.r.a monad on the category $\widehat{\mathbb{G}}$ of globular sets, and so applying it to internal categories, gives a 2-monad on the 2-category $\mathrm{CAT}(\widehat{\mathbb{G}})$ of globular categories which is familial and opfamilial.

7.4. Theorem. *If* $T : \mathcal{A} \to \mathcal{B}$ *is a familial 2-functor between finitely complete 2-categories and* $\phi : S \to T$ *is a cartesian 2-natural transformation then* S *is a familial 2-functor.*

Proof. By theorem(3.7) one has the diagram

$$\begin{tikzcd}
 & \mathrm{Spl}(T1) \arrow[r, "\phi_1^*"] \arrow[d, "U_{T1}"] \arrow[dl, phantom, "="] & \mathrm{Spl}(S1) \arrow[d, "U_{S1}"] \\
\mathcal{A} \arrow[ur, dotted] \arrow[r, shift right=2, "T_1"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right=60, "S_1"'] \arrow[rr, phantom, bend right=30, "\cong"] & \mathcal{B}/T1 \arrow[l, shift right=2, "L_T"'] \arrow[r, shift right=2, "\phi_1^*"'] \arrow[r, phantom, "\perp"] & \mathcal{B}/S1 \arrow[l, shift right=2, "\phi_{1!}"']
\end{tikzcd}$$

from which it is clear that S is familial. ■

7.5. Example. Recall the familial Fam_f from example(5.16). The monad $\mathcal{S}$ whose (strict) algebras are symmetric (strict) monoidal categories is a sub-monad of Fam_f. The category $\mathcal{S}(X)$ has the same objects as $\mathrm{Fam}_f(X)$, but a map

$$\begin{tikzcd}
n \arrow[rr, "f"] \arrow[dr, "a"'] & {} \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & m \arrow[dl, "b"] \\
& X &
\end{tikzcd}$$

is in $\mathcal{S}(X)$ iff $n = m$ and f is a bijection. One can easily check that Fam_f's monad structure restricts to $\mathcal{S}$, and so $\mathcal{S}$ is a cartesian 2-monad. The inclusion $\mathcal{S} \rightarrow \mathrm{Fam}_f$ is a cartesian 2-monad morphism and so by theorem(7.4) $\mathcal{S}$ is in fact a familial 2-monad.

7.6. Examples. Any club, as originally defined by Max Kelly in [Kelly, 1972], gives rise to a cartesian 2-monad morphism $T \rightarrow \mathcal{S}$ where T is a cartesian 2-monad, and the algebras of T coincide with the algebras of the original club. By theorem(7.4) T is a familial 2-monad. As a particular example, T could be the 2-monad for braided monoidal categories. Its explicit description is the same as $\mathcal{S}$'s above, except that f is a braid on n strings rather than a mere permutation. Yet another example, also obtainable from corollary(7.2), is the 2-monad for monoidal categories. As with this last case, we will see below that all of these examples are opfamilial also.

7.7. Theorem. *If $T : \mathcal{A} \rightarrow \mathcal{B}$ is a familial 2-functor between finitely complete 2-categories and $\mathbb{C}$ is a category then*

$$[\mathbb{C}, T] : [\mathbb{C}, \mathcal{A}] \rightarrow [\mathbb{C}, \mathcal{B}]$$

is a familial 2-functor.

Proof. Since $[\mathbb{C}, T](1)$ is constant at $T1$ and $[\mathbb{C}, \mathcal{B}]/[\mathbb{C}, T](1) \cong [\mathbb{C}, \mathcal{B}/T1]$, one can regard the canonical factorisation of $[\mathbb{C}, T]$ as given by the solid arrows in

$$\begin{tikzcd}
{[\mathbb{C}, \mathcal{A}]} \arrow[r, shift right=1.5, "{[\mathbb{C},T_1]}"'] \arrow[r, phantom, "\perp"] & {[\mathbb{C}, \mathcal{B}/T1]} \arrow[l, dotted, shift right=1.5, "{[\mathbb{C},L_T]}"'] \arrow[r, "{[\mathbb{C},t_{T1!}]}"] & {[\mathbb{C}, \mathcal{B}]}
\end{tikzcd}$$

and thus $[\mathbb{C}, T]$ is a parametric right 2-adjoint. Moreover since limits in $[\mathbb{C}, \mathcal{B}]$ are formed componentwise, the description of the fibrations monad can be made componentwise, and so $\mathrm{Spl}([\mathbb{C}, T](1)) \cong [\mathbb{C}, \mathrm{Spl}(T1)]$, and so $[\mathbb{C}, T]_1$ factors through $U_{[\mathbb{C},T](1)}$. ∎

7.8. Examples. For section(8) below and the general operad theory of [Weber, 2005] [Weber] we observe that for any category $\mathbb{C}$, and any of the familial 2-monads T among example(5.16) and examples(7.6), composition with T gives a familial 2-monad on $\mathrm{CAT}(\widehat{\mathbb{C}})$.

7.9. Theorem. *If $T : \mathcal{A} \rightarrow \mathcal{B}$ and $S : \mathcal{B} \rightarrow \mathcal{C}$ are familial 2-functors between finitely complete 2-categories then ST is familial.*

Proof. By theorem(6.6) we have

$$\begin{tikzcd}
& \mathrm{Spl}(T1) \arrow[r, "S_{T1}"] \arrow[d, "U_{T1}"] \arrow[dr, phantom, "="] & \mathrm{Spl}(ST1) \arrow[d, "U_{ST1}"] \\
\mathcal{A} \arrow[ur] \arrow[r, "T_1"'] \arrow[ur, phantom, "=", near end, shift left=3] & \mathcal{B}/T1 \arrow[r, "S_{T1}"'] & \mathcal{C}/ST1
\end{tikzcd}$$

and so $ST_1 = S_{T1}T_1$ factors through U_{ST1}. Since T is p.r.a T_1 has a left adjoint, and since S is p.r.a and by proposition(2.6), S_{T1} also has a left adjoint, and so ST is p.r.a. ■

7.10. Examples. Important for the theory higher symmetric operads [Weber, 2005] [Weber] are the examples of familial 2-functors that one obtains on $\mathrm{CAT}(\widehat{\mathbb{G}})$ by composing the familial 2-functors coming from examples(7.8) and examples(7.3).

In [Weber] analytic 2-monads on 2-toposes will be defined, and the operad theory of [Weber, 2005] will be developed further in this setting. The underlying 2-functor of an analytic 2-monad is in particular a familial 2-functor, and one of its additional properties is that evaluating it at 1 gives a groupoid. We now consider this extra condition. Let $T : \mathcal{A} \to \mathcal{B}$ be a 2-functor between finitely complete 2-categories, and suppose that $T1$ is a groupoid. Recall that by proposition(3.10) we have $\mathrm{Spl}(T1) \cong \mathrm{SplOp}(T1)$ commuting with the forgetful functors into $\mathcal{B}/T1$, and so we immediately obtain

7.11. Proposition. *Let $T : \mathcal{A} \to \mathcal{B}$ be a 2-functor between finitely complete 2-categories such that $T1$ is a groupoid. Then T is familial iff T is opfamilial.*

Since familial 2-functors T such that $T1$ is a groupoid are also opfamilial, they enjoy all the properties that familials were shown to in sections(6) and (7), and the corresponding dual properties. For instance, analytic 2-functors preserve split fibrations and split *op*fibrations. In addition we have

7.12. Theorem. *Let $T : \mathcal{A} \to \mathcal{B}$ be a familial 2-functor such that $T1$ is a groupoid.*

1. *T preserves groupoids.*
2. *T preserves lax pullbacks up to a left equivalence section.*
3. *T preserve lax pullbacks up to a right equivalence section.*

Proof. (1): let $T : \mathcal{A} \to \mathcal{B}$ be an analytic 2-functor between finitely complete 2-categories. If $G \in \mathcal{A}$ is a groupoid,

$$\begin{tikzcd}
X \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] \arrow[r, phantom, "\Downarrow\phi"] & TG
\end{tikzcd}$$

and let

$$\begin{tikzcd}
X \arrow[r, "g"] & TD \arrow[r, "Th"] & TG
\end{tikzcd}$$

be a T-generic factorisation for α_1. Then we have δ, ϕ_1 and ϕ_2 unique such that

$$\phi \quad = \quad \begin{tikzcd} X \arrow[r, "\alpha_2"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\phi_1}{\Nearrow}", near start] \arrow[dr, phantom, "\overset{T\phi_2}{\Nearrow}", near end] & TG \arrow[d, "T1_G"] \\ TD \arrow[ur, "T\delta" description] \arrow[r, "Th"'] & TG \end{tikzcd}$$

and ϕ_1 is chosen-Tt_G-cartesian. Since ϕ_1 is Tt_G-cartesian and $T1$ is a groupoid, ϕ_1 is invertible. Since G is a groupoid ϕ_2 is also invertible.
(2): we now refer to the proof of theorem(6.1)(1). The 2-cell ϕ_1 is chosen-$T(t_C)$-cartesian and thus invertible in this case since $T1$ is a groupoid, and so $\varepsilon : kr \rightarrow 1$ the counit of $k \dashv r$, is invertible.
(3): apply (2) to T^{co}. ∎

Proposition(7.1), corollary(7.2), theorem(7.7), and theorem(7.9) remain true after replacing familial 2-functors by those familial T such that $T1$ is a groupoid. With the exception of Fam and Fam_f, all the examples of familial 2-monads given so far all satisfy the groupoid condition. The only result of this section with no such refinement is theorem(7.4), which asserts that if $\phi : S \Rightarrow T$ is a cartesian 2-natural transformation and T is familial, then S is familial. The condition that $T1$ is a groupoid has no effect on whether $S1$ is a groupoid. For assume that $T1$ is a groupoid, and let $f : B \rightarrow T1$ be a morphism of $\mathcal{B}$. Then one can define a 2-functor S and a cartesian 2-natural transformation $\phi : S \Rightarrow T$ from the pullbacks

$$\begin{tikzcd} SA \arrow[r, "\phi_A"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TA \arrow[d, "Tt_A"] \\ B \arrow[r, "f"'] & T1 \end{tikzcd}$$

and by definition $S1 = B$.

7.13. Remark. We know that a familial 2-monad (T, η, μ) is cartesian: its functor part preserves pullbacks and the naturality squares of its unit and multiplication are pullbacks. If in addition $T1$ is a groupoid then (T, η, μ) is also cartesian in the obvious *bicategorical* sense. By theorem(7.12) the underlying 2-functor preserves bipullbacks. Moreover, the naturality squares of its unit and multiplication are bipullbacks. To see this for the multiplication (the proof for the unit is identical), note that for all $X \in \mathcal{K}$ that t_X is a split fibration and thus an isofibration, and so Tt_X is an isofibration by theorem(6.2). Thus the naturality square

$$\begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "T^2t_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "Tt_X"] \\ T^21 \arrow[r, "\mu_1"'] & T1 \end{tikzcd}$$

is a bipullback by example(3.12). Thus by proposition(3.13) all of the naturality squares of μ are bipullbacks.

8. Cartesianness of composite 2-monads

In [Weber, 2005] the situation was considered in which one has a cartesian monad S on CAT and a p.r.a monad T on $\widehat{\mathbb{G}}$ coming from a Batanin higher operad. One then regards both S and T as 2-monads on $\mathrm{CAT}(\widehat{\mathbb{G}})$: S is viewed as acting componentwise and T is viewed as acting on category objects in $\widehat{\mathbb{G}}$ because its functor part preserves pullbacks. Then theorem(7.12) of [Weber, 2005] says that there is a distributive law $\lambda : TS \to ST$ between S and T. The importance of this result is that for appropriate S and T, ST-operads are a symmetric analogue of the higher operads of [Batanin, 1998] and so are going to be appropriate for studying higher dimensional "weakly symmetric" monoidal structures and the stabilisation conjecture in the globular setting.

In this section we extend and in fact reprove theorem(7.12) of [Weber, 2005], so that the distributive laws obtained are seen to be cartesian, and so the composite monads that are obtained are familial. Moreover we work in an arbitrary presheaf category, replacing $\mathbb{G}$ by an arbitrary small category $\mathbb{C}$. So we let (T, η, μ) be a p.r.a monad on $\widehat{\mathbb{C}}$ as in sections(2) and (4), and use all the associated notation and results developed in those sections. We let S be a p.r.a 2-monad on CAT. As in the globular case we also regard S and T as 2-monads on $\mathrm{CAT}(\widehat{\mathbb{C}})$.

Recall from section(4) the monad $(\overline{T}, \overline{\eta}, \overline{\mu})$ on $\widehat{\Theta}_0$ and the monad morphism $(\widehat{\mathbb{C}}(i_0, 1), \varphi)$ from T to $\overline{T}$ established in proposition(4.8). We would also like to interpret this monad morphism in our 2-categorical setting, and so we must verify that $\overline{T}$ does indeed preserve pullbacks so that it can be applied to category objects. In fact we will now see that $\overline{T}$ is another p.r.a monad. To do this we first spell out in detail the adjunction $\mathrm{lan}_j \dashv \mathrm{res}_j$. For $X \in \widehat{\Theta}_0$ and $p \in \Theta_T$ we define

$$\mathrm{lan}_j(X)(p) = \sum_{f \in \mathrm{Kl}(T)(p,1)} XD_f \tag{8}$$

and we will justify this definition in lemma(8.1). Given a map $f : p \to 1$ in $\mathrm{Kl}(T)$ we denote by $c_f : XD_f \to \mathrm{lan}_j(X)(p)$ the corresponding coproduct inclusion. For $k : q \to p$ in Θ_T we define $\mathrm{lan}_j(X)(k)$ as the unique map such that

$$\begin{tikzcd}
XD_f \arrow[r, "c_f"] \arrow[d, "Xh_{f,k}"'] \arrow[dr, phantom, "="] & \mathrm{lan}_j(X)(p) \arrow[d, "\mathrm{lan}_j(X)(k)"] \\
XD_{fk} \arrow[r, "c_{fk}"'] & \mathrm{lan}_j(X)(q)
\end{tikzcd}$$

where $h_{f,k}$ is the unique free map so that

$$\begin{tikzcd}
q \arrow[r, "k"] \arrow[d, "g_{fk}"'] \arrow[dr, phantom, "="] & p \arrow[d, "g_f"] \arrow[ddr, bend left, "f"] & \\
D_{fk} \arrow[r, dashed, "h_{f,k}"] \arrow[ddr, phantom, "="] \arrow[drr, bend right, "t_{D_{fk}}"'] & D_f \arrow[dr, "t_{D_f}"] \arrow[r, phantom, "="] & {} \\
 & & 1
\end{tikzcd}$$

in $\mathrm{Kl}(T)$. Thus $\mathrm{lan}_j(X)$ is well-defined as an object of $\widehat{\Theta}_T$. Now for $X \in \widehat{\Theta}_0$ and $p \in \Theta_0$ we define $\overline{\eta}_{X,p} : Xp \to \mathrm{lan}_j(X)(p)$ to be the coproduct inclusion c_{t_p}. This makes sense because our choice of generic factorisations is normalised, and is clearly natural with respect to maps in Θ_0.

8.1. Lemma. *For $X \in \widehat{\Theta}_0$ the natural transformation*

$$\begin{tikzcd}
\Theta_0^{\mathrm{op}} \arrow[rr, "j^{\mathrm{op}}"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\overline{\eta}_X}{\Longrightarrow}"] & \Theta_T^{\mathrm{op}} \arrow[dl, "\mathrm{lan}_j(X)"] \\
 & \mathrm{Set} &
\end{tikzcd}$$

described above exhibits $\mathrm{lan}_j(X)$ as a pointwise left extension of X along j^{op}.

Proof. Since Set is cocomplete and Θ_0 and Θ_T are small it suffices to prove that $\overline{\eta}_X$ is a left extension. Given Z and ϕ as follows

$$\begin{tikzcd}
\Theta_0^{\mathrm{op}} \arrow[rr, "j^{\mathrm{op}}"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\phi}{\Longrightarrow}"] & \Theta_T^{\mathrm{op}} \arrow[dl, "Z"] \\
 & \mathrm{Set} &
\end{tikzcd}
=
\begin{tikzcd}
\Theta_0^{\mathrm{op}} \arrow[rr, "j^{\mathrm{op}}"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\overline{\eta}_X}{\Longrightarrow}"] & \Theta_T^{\mathrm{op}} \arrow[dl, "\Downarrow \hat{\phi}"] \arrow[dl, bend left=60, "Z"] \\
 & \mathrm{Set} &
\end{tikzcd}
\qquad (9)$$

we must define $\hat{\phi}$ unique such that equation(9) holds. For $p \in \Theta_T$ we define $\hat{\phi}_p$ as the unique function such that

$$\begin{tikzcd}
XD_f \arrow[r, "c_f"] \arrow[d, "\phi_{D_f}"'] \arrow[dr, phantom, "="] & \mathrm{lan}_j(X)(p) \arrow[d, "\hat{\phi}_p"] \\
ZD_f \arrow[r, "Zg_f"'] & Zp
\end{tikzcd}$$

for all $f : p \to 1$ in $\mathrm{Kl}(T)$. To see that this definition of $\hat{\phi}$ is natural in p let $k : q \to p$ be in Θ_T. We must show that for all $f : p \to 1$ in $\mathrm{Kl}(T)$ that $Z(k)\hat{\phi}_p c_f = \hat{\phi}_q \mathrm{lan}_j(X)(k) c_f$, and this follows from

$$\begin{tikzcd}
XD_f \arrow[rrr, "Xh_{f,k}"] \arrow[dr, "c_f"] \arrow[ddd, "\phi_{D_f}"'] & & & XD_{fk} \arrow[dl, "c_{fk}"'] \arrow[ddd, "\phi_{D_{fk}}"] \\
\arrow[r, phantom, "="] & \mathrm{lan}_j(X)(p) \arrow[r, "\mathrm{lan}_j(X)(k)"] \arrow[d, "\hat{\phi}_p"'] \arrow[u, phantom, "="'] & \mathrm{lan}_j(X)(q) \arrow[d, "\hat{\phi}_q"] & \arrow[l, phantom, "="'] \\
& Zp \arrow[r, "Zk"'] \arrow[d, phantom, "="] & Zq & \\
ZD_f \arrow[ur, "Zg_f"] \arrow[rrr, "Zh_{f,k}"'] & & & ZD_{fk} \arrow[ul, "Zg_{fk}"']
\end{tikzcd}$$

the outside of which also commutes. By definition for $p \in \Theta_0$ we have $\hat{\phi}_p c_{t_p} = \phi_p$ and so $\hat{\phi}$ satisfies equation(9). Finally we verify the uniqueness of $\hat{\phi}$. Notice that for all $p \in \Theta_T$ and $f : p {\rightarrow} 1$ in $\mathrm{Kl}(T)$, we have

$$\mathrm{lan}_j(X)(g_f)c_{t_{D_f}} = c_f \tag{10}$$

by the definition of the arrow map of $\mathrm{lan}_j(X)$. Thus all the regions of

$$\begin{tikzcd}
XD_f \arrow[drr, bend left, "\phi_{D_f}"] \arrow[dr, "c_{t_{D_f}}"] \arrow[ddr, bend right, "c_f"'] & & \\
& \mathrm{lan}_j(X)(D_f) \arrow[r, "\hat{\phi}_{D_f}"] \arrow[d, "\mathrm{lan}_j(X)(g_f)"] & ZD_f \arrow[d, "Zg_f"] \\
& \mathrm{lan}_j(X)(p) \arrow[r, "\hat{\phi}_p"'] & Zp
\end{tikzcd}$$

commute, and so the definition of $\hat{\phi}$ is forced by equation(9) and the required naturality of $\hat{\phi}$. ■

By lemma(8.1) the following description of lan_j's arrow maps is forced, and with respect to these arrow maps $\overline{\eta}_X$ is natural in X. For $\psi : X {\rightarrow} X'$ in $\widehat{\Theta}_0$ and $p \in \Theta_0$, the function $\mathrm{lan}_j(\psi)_p$ is unique such that

$$\begin{tikzcd}
XD_f \arrow[r, "c_f"] \arrow[d, "\psi_{D_f}"'] & \mathrm{lan}_j(X)(p) \arrow[d, "\mathrm{lan}_j(\psi)_p"] \\
X'D_f \arrow[r, "c_f"'] & \mathrm{lan}_j(X')(p)
\end{tikzcd}$$

commutes for all $f : p {\rightarrow} 1$ in $\mathrm{Kl}(T)$. Now for $Z \in \widehat{\Theta}_T$ and $p \in \Theta_T$ we define the function

$$\overline{\varepsilon}_{Z,p} : \mathrm{lan}_j(Zj^{\mathrm{op}})(p) \rightarrow Zp$$

to be unique such that $\overline{\varepsilon}_{X,p} c_f = Xg_f$ for all $f : p {\rightarrow} 1$ in $\mathrm{Kl}(T)$.

8.2. Lemma. *The functions $\overline{\varepsilon}_{Z,p}$ just defined are natural in Z and p, and the resulting $\overline{\varepsilon}$ together with $\overline{\eta}$ form the counit and unit of an adjunction* $\mathrm{lan}_j \dashv \mathrm{res}_j$.

Proof. To verify naturality in p let $k : q \to p$ be in Θ_T. For each $f : p \to 1$ in $\mathrm{Kl}(T)$ we have

$$\begin{tikzcd}
ZD_f \arrow[r, "c_f"] \arrow[d, "Zh_{f,k}"'] \arrow[rr, bend left=40, "Zg_f"] \arrow[rr, phantom, bend left=20, "="] \arrow[dr, phantom, "="] & \mathrm{lan}_j(Zj^{\mathrm{op}})(p) \arrow[r, "\overline{\varepsilon}_{Z,p}"] \arrow[d, "\mathrm{lan}_j(Zj^{\mathrm{op}})(k)"] & Zp \arrow[d, "Zk"] \\
ZD_{fk} \arrow[r, "c_{fk}"] \arrow[rr, bend right=40, "Zg_{fk}"'] \arrow[rr, phantom, bend right=20, "="] & \mathrm{lan}_j(Zj^{\mathrm{op}})(q) \arrow[r, "\overline{\varepsilon}_{Z,q}"] & Zq
\end{tikzcd}$$

in which the outside also commutes, and so $Z(k)\overline{\varepsilon}_{Z,p}c_f = \overline{\varepsilon}_{Z,q}\mathrm{lan}_j(Zj^{\mathrm{op}})c_f$ as required. To verify naturality in Z let $\psi : Z \to Z'$ in $\widehat{\Theta}_T$, then for all $f : p \to 1$ in $\mathrm{Kl}(T)$ we have

$$\begin{tikzcd}
ZD_f \arrow[r, "c_f"] \arrow[d, "\psi_{D_f}"'] \arrow[rr, bend left=40, "Zg_f"] \arrow[rr, phantom, bend left=20, "="] \arrow[dr, phantom, "="] & \mathrm{lan}_j(Zj^{\mathrm{op}})(p) \arrow[r, "\overline{\varepsilon}_{Z,p}"] \arrow[d, "\mathrm{lan}_j(\psi j^{\mathrm{op}})_p"] & Zp \arrow[d, "\psi_p"] \\
Z'D_f \arrow[r, "c_{fk}"] \arrow[rr, bend right=40, "Z'g_f"'] \arrow[rr, phantom, bend right=20, "="] & \mathrm{lan}_j(Z'j^{\mathrm{op}})(p) \arrow[r, "\overline{\varepsilon}_{Z',p}"] & Z'p
\end{tikzcd}$$

in which the outside also commutes, and so $\psi_p\overline{\varepsilon}_{Z,p}c_f = \overline{\varepsilon}_{Z',q}\mathrm{lan}_j(\psi j^{\mathrm{op}})c_f$ as required. We already know from lemma(8.1) that $\overline{\eta}$ is the unit of an adjunction $\mathrm{lan}_j \dashv \mathrm{res}_j$, and so by example(2.17) of [Weber, 2007] for instance, it suffices to show that one of the triangular identities of an adjunction are satisfied by $\overline{\varepsilon}$ and $\overline{\eta}$. To this end notice that for $p \in \Theta_T$ that $\overline{\varepsilon}_{Z,p}c_{t_p} = Z(g_{t_p})$ by definition, and so $\mathrm{res}_j\overline{\varepsilon} \circ \overline{\eta}\mathrm{res}_j = \mathrm{id}$ as required. ■

8.3. Corollary. *$(\overline{T}, \overline{\eta}, \overline{\mu})$ is a p.r.a monad.*

Proof. By theorem(2.13) and the explicit description of lan_j given above $\overline{T}$ is p.r.a. Since presheaf categories are small-extensive $\overline{\eta}$ is a cartesian transformation by definition. To see that $\overline{\mu}$ is cartesian it suffices, by elementary properties of pullback squares, to show that

$$\begin{tikzcd}
\overline{T}^2(X)(p) \arrow[r, "\overline{\mu}_{X.p}"] \arrow[d, "\overline{T}^2(t_X)_p"'] & \overline{T}(X)(p) \arrow[d, "\overline{T}(t_X)_p"] \\
\overline{T}^2(1)(p) \arrow[r, "\overline{\mu}_{1,p}"'] & \overline{T}(1)(p)
\end{tikzcd}$$

is a pullback for all X and p. An element of $\overline{T}^2(1)(p)$ is a pair $(f : p\longrightarrow 1, f_2 : D_f\longrightarrow 1)$ of maps in $\mathrm{Kl}(T)$. Given these maps we have

$$\begin{tikzcd}
\overline{T}(X)(D_f) \arrow[r, "c_f"] \arrow[dr, "\overline{T}(X)(g_f)"'] & \overline{T}^2(X)(p) \arrow[d, "\overline{\mu}_{X,p}"] \arrow[dl, phantom, "="] \\
 & \overline{T}(X)(p)
\end{tikzcd}
\qquad
\begin{tikzcd}
XD_{f_2} \arrow[r, "Xh_{f_2,g_f}"] \arrow[d, "c_{f_2}"'] \arrow[dr, phantom, "="] & XD_{f_2,g_f} \arrow[d, "c_{f_2,g_f}"] \\
\overline{T}(X)(D_f) \arrow[r, "\overline{T}(X)(g_f)"'] & \overline{T}(X)(p)
\end{tikzcd}$$

where h_{f_2,g_f} is the unique free map such that

$$\begin{tikzcd}
p \arrow[r, "g_f"] \arrow[d, "g_{f_2g_f}"'] \arrow[dr, phantom, "="] & D_f \arrow[d, "f_2"] \arrow[ddr, bend left, "f_2"] & \\
D_{f_2g_f} \arrow[r, dotted, "h_{f_2,g_f}"] \arrow[drr, bend right, "t_{D_{f_2g_f}}"'] & D_{f_2} \arrow[dr, "t_{D_{f_2}}"] \arrow[r, phantom, "="] \arrow[d, phantom, "="] & {} \\
 & {} & 1
\end{tikzcd}$$

and $\overline{\mu}_{1,p}(f, f_2) = f_2g_f$. An element α of $\overline{T}(X)(p)$ which is sent to f_2g_f by $\overline{T}^2(t_X)_p$ is by definition an element of XD_{f_2,g_f}. Since $g_{f_2}g_f$ is generic, h_{f_2,g_f} is invertible, and so there is a unique element β such that $Xh_{f_2,g_f}(\beta) = \alpha$. Now β regarded as an element of $\overline{T}^2(X)(p)$, is unique such that $\overline{T}^2(t_X)_p(\beta) = (f, f_2)$ and $\overline{\mu}_{X,p}(\beta) = \alpha$. ■

Thus as promised we have a monad morphism

$$(\mathrm{CAT}(\widehat{\mathbb{C}})(i_0, 1), \varphi) : (\mathrm{CAT}(\widehat{\mathbb{C}}), T) \rightarrow (\mathrm{CAT}(\widehat{\Theta}_0), \overline{T})$$

in the 2-category of 2-categories.

Let S be a 2-monad on CAT whose underlying 2-functor is p.r.a, and we regard S and $\overline{T}$ as 2-monads on $\mathrm{CAT}(\widehat{\Theta}_0)$. We now describe a cartesian distributive law $\overline{\lambda} : \overline{T}S{\longrightarrow}S\overline{T}$. For $X \in \mathrm{CAT}(\widehat{\Theta}_0)$ and $p \in \Theta_0$ define the functor $\overline{\lambda}_{X,p}$ as the unique functor such that

$$\begin{tikzcd}
S(XD_f) \arrow[r, "c_f"] \arrow[dr, "Sc_f"'] & \overline{T}(S{\circ}X)(p) \arrow[d, "\overline{\lambda}_{X,p}"] \arrow[dl, phantom, "="] \\
 & S(\overline{T}(X)(p))
\end{tikzcd}$$

8.4. Lemma. *The functors $\overline{\lambda}_{X.p}$ just defined are the components of a cartesian transformation $\overline{\lambda} : \overline{T}S \rightarrow S\overline{T}$.*

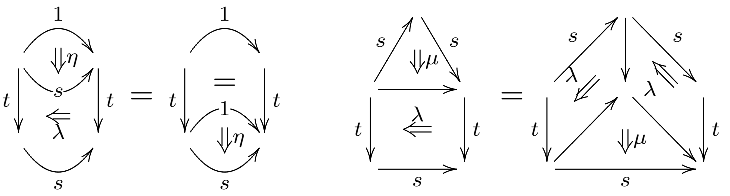
Proof. To verify that the above definition is natural in p let $k : q \rightarrow p$ be in Θ_0. Then the desired naturality follows since for all $f : p \rightarrow 1$ in $\mathrm{Kl}(T)$ we have

$$\begin{tikzcd}
S(XD_f) \arrow[r, "c_f"] \arrow[d, "S(Xh_{f,k})"'] \arrow[rr, bend left=40, "Sc_f"] \arrow[dr, phantom, "="] & \overline{T}(S{\circ}X)(p) \arrow[r, "\overline{\lambda}_{X,p}"] \arrow[d, "\overline{T}(S{\circ}X)(k)"] \arrow[u, phantom, "="] & S(\overline{T}(X)(p)) \arrow[d, "S(\overline{T}(X)(k))"] \\
S(XD_{fk}) \arrow[r, "c_{fk}"] \arrow[rr, bend right=40, "Sc_{fk}"'] & \overline{T}(S{\circ}X)(q) \arrow[r, "\overline{\lambda}_{X,q}"] \arrow[d, phantom, "="] & S(\overline{T}(X)(p)) \\
 & &
\end{tikzcd}$$

where the outside also commutes. By definition $\overline{\lambda}_{X,p}$ is 2-natural in X. It remains to see that it is in fact cartesian natural in X. Notice that $\overline{\lambda}_{X,p}$ is a coproduct comparison map, and so by lemma(2.16) the result follows. ∎

We must now verify that $\overline{\lambda}$ satisfies the axioms of a distributive law. Instead of verifying the axioms directly we exploit instead the correspondence between distributive laws and monad liftings. First we recall this correspondence.

Given a 2-category $\mathcal{K}$ and monad (A, t) in $\mathcal{K}$, an Eilenberg-Moore object consists of an object A^t of $\mathcal{K}$ together with a monad morphism $(u, \tau) : (A^t, 1) \rightarrow (A, t)$ satisfying a universal property[16]. For instance in CAT the object A^t is the category of algebras of the monad t, and u is the forgetful functor. As one would expect from this basic case, the forgetful arrow u has a left adjoint f and $uf = t$. Given monads (A, s) and (A, t) in $\mathcal{K}$ a distributive law of s over t consists of $\lambda : ts \rightarrow st$ such that (s, λ) is a monad morphism $(A, t) \rightarrow (A, t)$ and



In other words, λ satisfies 4 axioms which express its compatibility with the units and multiplications of s and t. Let $\mathcal{K}$ be a 2-category with Eilenberg-Moore objects. To give a distributive law $\lambda : ts \rightarrow st$ in $\mathcal{K}$ is the same as giving a lifting of the monad s to a monad

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[16] Eilenberg-Moore objects are a special kind of 2-categorical limit. In particular note that by [Street, 1976] any finitely complete 2-category has Eilenberg-Moore objects. See [Street, 1972] [Lack-Street, 2002] for further discussion.

$\overline{s}$ on A^t, the object of t-algebras, in the sense that we have a monad $(\overline{s}, \overline{\eta}, \overline{\mu})$ on A^t such that $su = u\overline{s}$ and

$$\begin{tikzcd}
A^T \arrow[r, bend left, "1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\overline{\eta}"] \arrow[d, "u"'] & A^T \arrow[d, "u"] & & A^T \arrow[r, bend left, "\overline{s}^2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\overline{\mu}"] \arrow[d, "u"'] & A^T \arrow[d, "u"] \\
A \arrow[r, bend left] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\eta"] & A & & A \arrow[r, bend left] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\mu"] & A
\end{tikzcd}$$

commute. Given a lifting $\overline{s}$ as above, one obtains the corresponding distributive law as the composite

$$ts \xrightarrow{ts\eta} tst \xrightarrow{u\varepsilon\overline{s}f} st \; .$$

In our situation $\mathcal{K}$ is the 2-category of 2-categories, 2-functors and 2-natural transformations. Monads in $\mathcal{K}$ are 2-monads and the object A^t of algebras of a 2-monad t is the 2-category t-Alg_s. Observe that the 2-monad on $\mathrm{CAT}(\widehat{\Theta}_0)$ given by composition with S lifts to a monad on $\mathrm{CAT}(\widehat{\Theta}_T)$ which by lemma(4.5) may be identified with $\overline{T}$-Alg_s, and so corresponds to a distibutive law, and we shall now see that this distributive law is given by $\overline{\lambda}$. To see this note that for all $f : p \longrightarrow 1$ in $\mathrm{Kl}(T)$ we have

$$\begin{tikzcd}
\overline{T}(S\circ X)(p) \arrow[r, "\overline{T}(S\circ\overline{\eta}_X)_p"] \arrow[dr, phantom, "="] & \overline{T}(S\circ\overline{T}X)(p) \arrow[ddr, bend left, "\overline{\varepsilon}_{S\circ\overline{T}X,p}"] & \\
S(XD_f) \arrow[u, "c_f"'] \arrow[r, "Sc_{t_{D_f}}"] \arrow[drr, bend right, "c_f"'] & S(\overline{T}(X)(D_f\overline{\overline{)}})) \arrow[u, "c_f"'] \arrow[dr, "S(\overline{T}X(g_f))"] \arrow[d, phantom, "(I)"] & \\
 & & S(\overline{T}(X)(p))
\end{tikzcd}$$

and we note that region (I) also commutes by by equation(10) in the proof of lemma(9). By the definition of $\overline{\lambda}$ this yields $\overline{\lambda}_{X,p} = \overline{\varepsilon}_{S\circ\overline{T}X,p}\overline{T}(S\circ\overline{\eta}_X)_p$ and so $\overline{\lambda}$ is indeed the distributive law corresponding to the canonical lifting of S to $\mathrm{CAT}(\widehat{\Theta})$ as claimed. We have proved:

8.5. Corollary. *Let T be a p.r.a monad on $\widehat{\mathbb{C}}$ and S be a 2-monad on* CAT *whose underlying 2-functor is p.r.a. Then the functors $\overline{\lambda}_{X.p}$ defined above are the components of a cartesian distributive law $\overline{\lambda} : \overline{T}S \longrightarrow S\overline{T}$ between the 2-monads S and $\overline{T}$ on* $\mathrm{CAT}(\widehat{\Theta}_0)$.

Now we assume that T-cardinals are connected. We shall now explain how $\overline{\lambda}$ restricts to a cartesian distributive law $\lambda : TS \longrightarrow ST$. Let $p \in \Theta_0$ and recall its description as a canonical colimit of representables that we described prior to lemma(4.12). For $x : C \longrightarrow p$ in y/p and $X \in \mathrm{CAT}(\widehat{\mathbb{C}})$ we write $\pi_{X,x}$ for the composite functor

$$\mathrm{CAT}(\widehat{\mathbb{C}})(p, X) \xrightarrow{-\circ x} \mathrm{CAT}(\widehat{\mathbb{C}})(C, X) \cong XC$$

and note that the $\pi_{X,x}$ for $x : C \longrightarrow p$, form the components of a limit cone. Define

$$\kappa_{X,p} : S(\mathrm{CAT}(\widehat{\mathbb{C}})(p, X)) \to \mathrm{CAT}(\widehat{\mathbb{C}})(p, S\circ X)$$

as the unique functor such that $\pi_{S\circ X,x}\kappa = S(\pi_{X,x})$ for all $x : C \longrightarrow p$.

8.6. Lemma. *The functors $\kappa_{X,p}$ just described are the components of a natural isomorphism.*

Proof. By definition $\kappa_{X,p}$ is natural in p and 2-natural in X, and is the comparison map for a connected limit since T-cardinals are connected. It is an isomorphism since S as a p.r.a 2-functor preserves connected limits. ∎

8.7. Lemma. *κ is the 2-cell part of a monad morphism*

$$(\mathrm{CAT}(\widehat{\mathbb{C}})(i_0,1),\kappa) : (S,\mathrm{CAT}(\widehat{\mathbb{C}}))\to(S,\mathrm{CAT}(\widehat{\Theta}_0))$$

Proof. To see that κ is compatible with the unit note that for all $x : C\to p$ we have

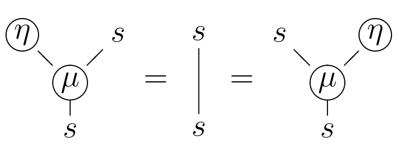
$$\begin{tikzcd}
\mathrm{CAT}(\widehat{\mathbb{C}})(p,X) \arrow[r, "\pi"] \arrow[d, "\eta"'] & XC \arrow[d, "\eta"] \\
S(\mathrm{CAT}(\widehat{\mathbb{C}})(p,X)) \arrow[r, "S\pi"] \arrow[d, "\kappa"'] & SXC \\
\mathrm{CAT}(\widehat{\mathbb{C}})(p,S\circ X) \arrow[ur, "\pi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{CAT}(\widehat{\mathbb{C}})(p,X) \arrow[r, "\pi"] \arrow[d, "\mathrm{CAT}(\widehat{\mathbb{C}})(p,\eta)"'] & XC \arrow[d, "\eta"] \\
S\mathrm{CAT}(\widehat{\mathbb{C}})(p,X) \arrow[r, "\pi"'] & SXC
\end{tikzcd}$$

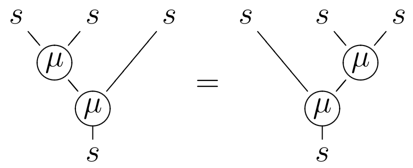
in which all the regions are commutative by the naturality of η, the definition of κ and the naturality of $\pi_{X,x}$ in X. Since the $\pi_{S\circ X,x}$ form limit cones the result follows. To see that κ is compatible with the multiplication note that for all $x : C\to p$ we have

$$\begin{tikzcd}
S^2(\mathrm{CAT}(\widehat{\mathbb{C}})(p,X)) \arrow[r, "S^2\pi"] \arrow[d, "S\kappa"'] & S^2X(C) \arrow[d, "\mu"] \\
S(\mathrm{CAT}(\widehat{\mathbb{C}})(p,S\circ X)) \arrow[ur, "S\pi"] \arrow[d, "\kappa"'] & SX(C) \\
\mathrm{CAT}(\widehat{\mathbb{C}})(p,S^2\circ X) \arrow[uur, "\pi"'] \arrow[r, "\mathrm{CAT}(\widehat{\mathbb{C}})(p,\mu)"'] & \mathrm{CAT}(\widehat{\mathbb{C}})(p,S\circ X) \arrow[u, "\pi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2(\mathrm{CAT}(\widehat{\mathbb{C}})(p,X)) \arrow[d, "\mu"'] \arrow[dr, "S^2\pi"] & \\
S(\mathrm{CAT}(\widehat{\mathbb{C}})(p,X)) \arrow[d, "\kappa"'] \arrow[dr, "S\pi"] & S^2X(C) \arrow[d, "\mu"] \\
\mathrm{CAT}(\widehat{\mathbb{C}})(p,S\circ X) \arrow[r, "\pi"'] & SX(C)
\end{tikzcd}$$

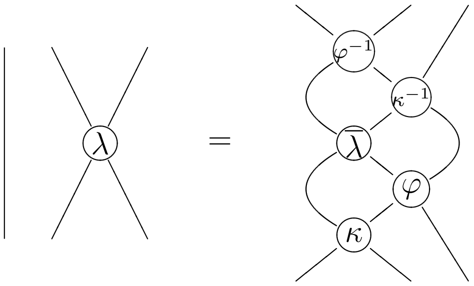
in which all the regions are commutative by the naturality of μ, the definition of κ and the naturality of $\pi_{X,x}$ in X. Since the $\pi_{S\circ X,x}$ form limit cones the result follows. ∎

To describe λ, we shall use the string diagrams of [Joyal-Street, 1991] to depict arrows and 2-cells in a 2-category. As an illustration, the axioms for a monad (A,s) are depicted as





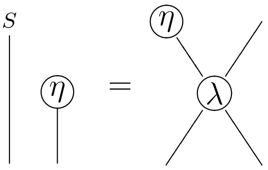
so our string diagrams go from top to bottom, and we always use η to denote the unit of a monad, and μ the multiplication. In subsequent diagrams we shall omit labels on the strings when no ambiguity results. The reader will easily translate the axioms for a monad morphism and for a distributive law into the language of string diagrams. Now λ is defined to be to be the unique 2-natural transformation such that

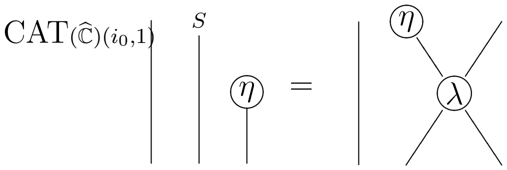


The left hand side of the above equation is $\mathrm{CAT}(\widehat{\mathbb{C}})(i_0,1)\lambda$, and so this definition makes sense since $\mathrm{CAT}(\widehat{\mathbb{C}})(i_0,1)$ is fully faithful.

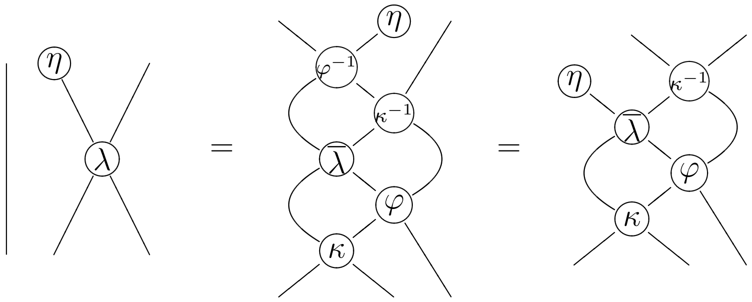
8.8. Theorem. *Let T be a p.r.a monad on $\widehat{\mathbb{C}}$ such that T-cardinals are connected, and let S be a 2-monad on* CAT *whose underlying 2-functor is p.r.a. Then $\lambda : TS \to ST$ defined above is a cartesian distributive law.*

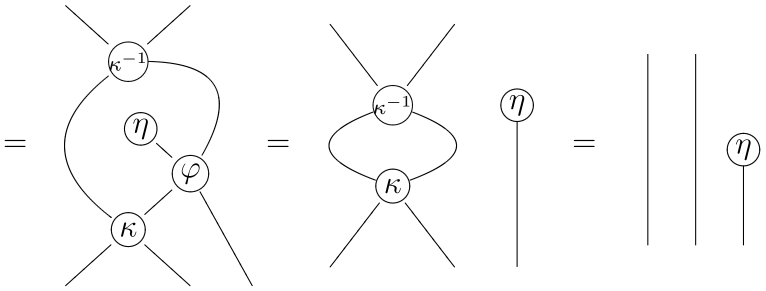
Proof. It is cartesian since $\overline{\lambda}$ is cartesian and $\mathrm{CAT}(\widehat{\mathbb{C}})(i_0,1)$, as a fully faithful right adjoint, reflects pullbacks. We now verify the 4 distributive law axioms. The axiom which expresses the compatibility of λ with T's unit is given on the left



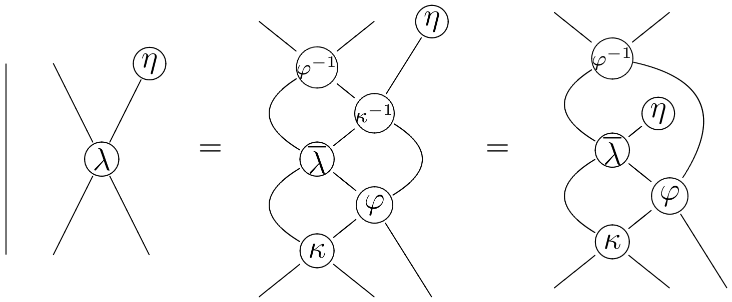


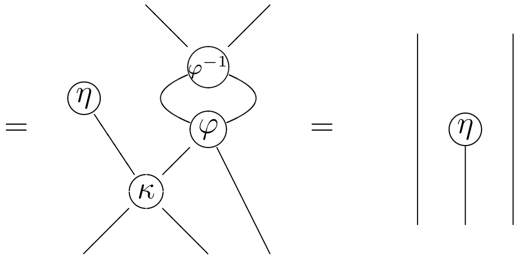
but it suffices to verify this axiom after composition with $\mathrm{CAT}(\widehat{\mathbb{C}})(i_0,1)$ because this 2-functor is fully faithful, in other words, it suffices to verify the axiom given on the right in the previous display. Similarly with the other three distibutive law axioms: we verify them after composition with $\mathrm{CAT}(\widehat{\mathbb{C}})(i_0,1)$. Proceeding in this way we verify λ's compatibility with T's unit,



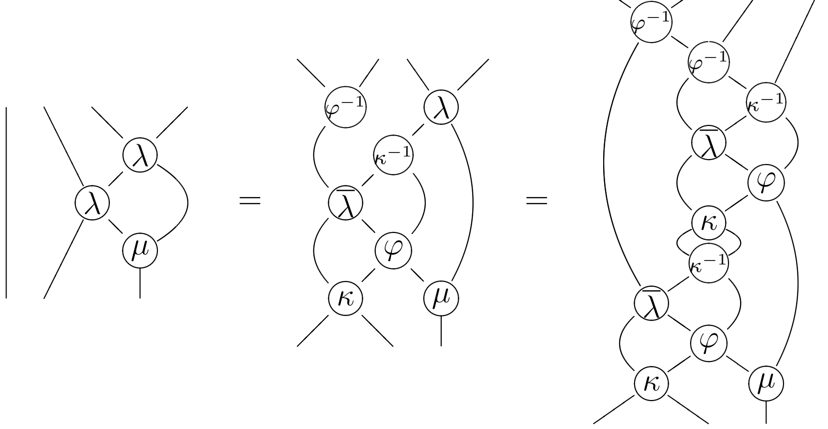


λ's compatibility with S's unit,





λ's compatibility with T's multiplication,



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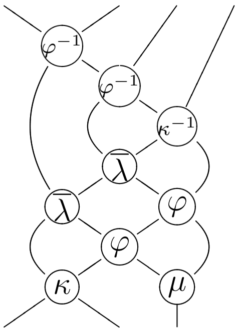

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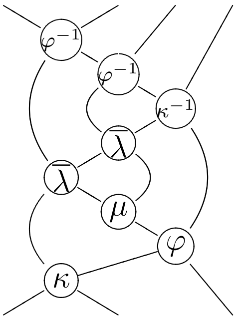

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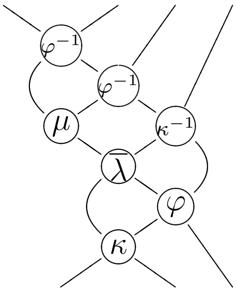

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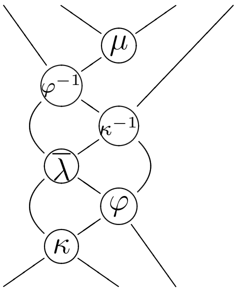

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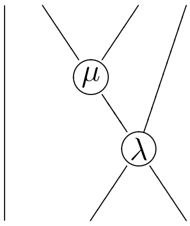
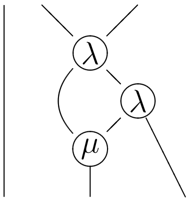


and finally λ's compatibility with S's multiplication,



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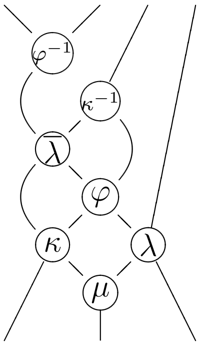

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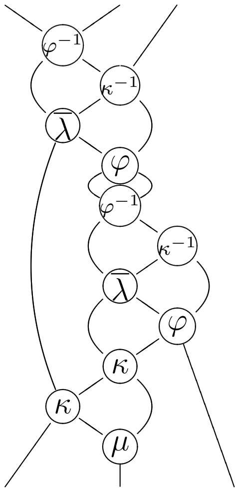

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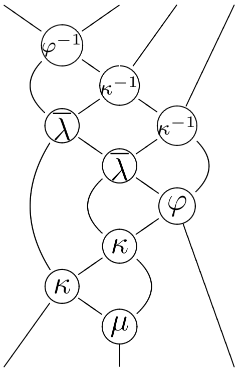

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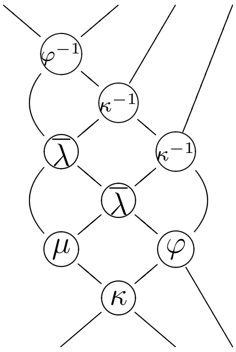

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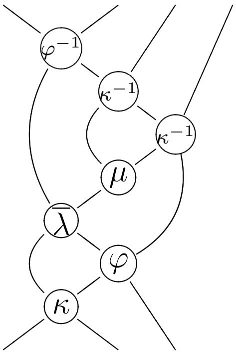

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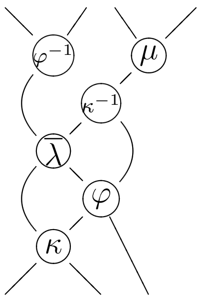

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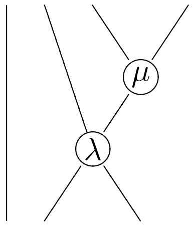

and so λ is indeed a distributive law. ■

Since the monad structure on ST is constructed using the monad structures of S and T and the distributive law established in theorem(8.8), by corollary(7.2) we immediately obtain the following result.

8.9. Corollary. *Let T be a p.r.a monad on $\widehat{\mathbb{C}}$ such that T-cardinals are connected.*

1. *If S is a p.r.a 2-monad on* CAT *then the 2-functor ST on* $\mathrm{CAT}(\widehat{\mathbb{C}})$ *has the structure of a p.r.a 2-monad.*
2. *If S is a familial (resp. opfamilial) 2-monad on* CAT *then the 2-functor ST on* $\mathrm{CAT}(\widehat{\mathbb{C}})$ *has the structure of a familial (resp. opfamilial) 2-monad. Moreover if $S1$ is a groupoid then so is $ST1$.*

8.10. Examples. A monad T on $\widehat{\mathbb{G}}$ arising from an ω-operad of [Batanin, 1998] is a p.r.a monad and T-cardinals, being globular cardinals, are connected. Thus the 2-monads of [Weber, 2005] giving rise to symmetric and braided analogues of higher operads are all familial.

References

J. Adamek and J. Rosický, Locally presentable and accessible categories, LMS Lecture Notes No 189, Cambridge University Press, 1994.

M. Batanin, Monoidal globular categories as a natural environment for the theory of weak n-categories, Advances in Mathematics, 136:39–103, 1998.

M. Batanin, The Eckmann-Hilton argument and higher operads, to appear in Adv. Math, arXiv:math.CT/0207281, 2002.

M. Batanin, Symmetrisation of n-operads and compactification of real configuration spaces, to appear in Adv. Math. arXiv:math.CT/0606067, 2006.

M. Batanin, A. Joyal, J. Kock, and J. F. Mascari, Polynomial functors and opetopes, in preparation.

J. Bénabou, Fibred categories and the foundation of naive category theory, Journal of Symbolic Logic, 50:10–37, 1985.

J. Bénabou, Some remarks on free monoids in a topos, Lecture Notes in Math, 1488:20–29, 1991.

C. Berger, A cellular nerve for higher categories, Advances in Mathematics, 169:118–175, 2002.

R. Blackwell, G. M. Kelly, and A. J. Power, Two-dimensional monad theory, J. Pure Appl. Algebra, 59:1–41, 1989.

A. Carboni and P. T. Johnstone, Connected limits, familial representability and Artin glueing, Mathematical Structures in Computer Science, 5:441–459, 1995.

A. Carboni, S. Lack, and R. F. C. Walters, Introduction to extensive and distributive categories, J. Pure Appl. Algebra, 84:145–158, 1993.

F. Conduché, Au aujet de l'existence d'adjoints à droite aux foncteurs "image réciproque" dans la catégory des catégories. C. R. Acad. Sci. Paris Sér.A-B, 275:A891–A894, 1972.

Y. Diers. Catégories Localisables, PhD thesis, Université de Paris VI, 1977.

E. Dubuc, Kan extensions in enriched category theory, Lecture Notes in Math 145, Springer Verlag, 1970.

P. Freyd, Aspects of topoi, Bull. Austral. Math. Soc, 7:1–76, 1972.

N. Gambino and M. Hyland, Well-founded trees and dependent polynomial functors, in Types 2003, volume 3085, pages 210–225, 2004.

J. Giraud, Méthode de la descente, Bull. Soc. Math. France Mém, 2, 1972.

B. Jacobs, Categorical logic and type theory, Studies in logic and the foundations of mathematics no 141, Elsevier, 1999.

M. Johnson, Pasting diagrams in n-categories with applications to coherence theorems and categories of paths, PhD thesis, University of Sydney, 1987.

A. Joyal, Disks, duality and θ-categories, unpubished note, 1997.

A. Joyal and R. Street, The geometry of tensor calculus I, Advances in Mathematics, 88:55–112, 1991.

G.M. Kelly, An abstract approach to coherence, Lecture Notes in Math, 281:106–147, 1972.

J. Kock, Notes on polynomial functors (draft), available at http://mat.uab.es/˜ kock/cat.html.

S. Lack and R. H. Street, The formal theory of monads II, J. Pure Appl. Algebra, 175:243–265, 2002.

T. Leinster, Nerves of algebras, talk at CT04 in Vancouver, available at http://www.maths.gla.ac.uk/˜ tl/vancouver/.

I. Moerdijk and E. Palmgren, Wellfounded trees in categories, Annals of Pure and Applied Logic, 104:189–218, 2000.

I. Moerdijk and I. Weiss, Dendroidal sets, Algebraic and Geometric Topology, 7:1441–1470, 2007a.

I. Moerdijk and I. Weiss, On inner Kan complexes in the category of Dendroidal sets, available at arXiv:math.AT/0701295, 2007b.

R. Paré, Absolute coequalizers, Lecture Notes in Math, 86:132–145, 1969.

R. Street, The formal theory of monads, J. Pure Appl. Algebra, 2:149–168, 1972.

R. Street, Fibrations and Yoneda's lemma in a 2-category, Lecture Notes in Math, 420:104–133, 1974.

R. Street, Limits indexed by category-valued 2-functors, J. Pure Appl. Algebra, 8:149–181, 1976.

R. Street, Cosmoi of internal categories, Trans. Amer. Math. Soc, 258:271–318, 1980a.

R. Street, Fibrations in bicategories, Cahiers Topologie Géom. Differentielle, 21:111–160, 1980b.

R. Street, The petit topos of globular sets, J. Pure Appl. Algebra, 154:299–315, 2000.

R. Street and R. F. C. Walters, Yoneda structures on 2-categories, J.Algebra, 50:350–379, 1978.

M. Weber, Operads within monoidal pseudo algebras II, in preparation.

M. Weber, Generic morphisms, parametric representations, and weakly cartesian monads, Theory and applications of categories, 13:191–234, 2004.

M. Weber, Operads within monoidal pseudo algebras, Applied Categorical Structures, 13:389–420, 2005.

M. Weber, Yoneda structures from 2-toposes, Applied Categorical Structures, 15:259–323, 2007.

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