Faà di Bruno for operads and internal algebras

Joachim Kock and Mark Weber

Abstract. For any coloured operad R, we prove a Faà di Bruno formula for the 'connected Green function' in the incidence bialgebra of R. This generalises on one hand the classical Faà di Bruno formula (dual to composition of power series), corresponding to the case where R is the terminal reduced operad, and on the other hand the Faà di Bruno formula for P-trees of Gálvez–Kock–Tonks (P a finitary polynomial endofunctor), which corresponds to the case where R is the free operad on P. Following Gálvez–Kock–Tonks, we work at the objective level of groupoid slices, hence all proofs are 'bijective': the formula is established as the homotopy cardinality of an explicit equivalence of groupoids. In fact we establish the formula more generally in a relative situation, for algebras for one polynomial monad internal to another. This covers in particular nonsymmetric operads (for which the terminal reduced case yields the noncommutative Faà di Bruno formula of Brouder–Frabetti–Krattenthaler).

Contents

0. Background

The Faà di Bruno formula computes the coefficients of the composite of two formal power series without constant terms. For

$$f(z) = \sum_{n=1}^{\infty} f_n \frac{z^n}{n!} \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} g_n \frac{z^n}{n!}$$

the composite series

$$(g \circ f)(z) =: \sum_{n=1}^{\infty} h_n \frac{z^n}{n!} \qquad \text{has} \qquad h_n = \sum_{k=1}^{n} B_{n,k}(f_1, f_2, \ldots) \cdot g_k$$

2010 *Mathematics Subject Classification.* 16T10, 05A19, 18D50, 18C15, 18B40, 32A05, 57T30, 18G30.

where $B_{n,k}(x_1, x_2, x_3, \ldots)$ are the partial Bell polynomials, whose coefficient of a monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ counts the number of partitions of an n-element set into k blocks: λ_1 blocks of size 1, λ_2 blocks of size 2, etc. ($\sum_i \lambda_i = k$ and $\sum_i i\lambda_i = n$).

We shall not actually need this classical formulation, for which we refer to the survey of Figueroa and Gracia-Bondía [**18**]. An excellent historical account of the formula is due to Johnson [**29**].

Our starting point is instead an elegant coalgebraic rendition of the formula, which, as far as we know, was first noticed by Brouder, Frabetti and Krattenthaler [**6**], inspired by constructions in perturbative quantum field theory. Let A_k denote the linear functional on the vector space of power series that takes f to $f^{(k)}(0)$, i.e. returns the coefficient of $z^k/k!$. The *Faà di Bruno bialgebra* is the polynomial ring $\mathbb{C}[A_1, A_2, A_3, \ldots]$ with comultiplication dual to the monoid structure of composition of power series. Doubilet [**14**] proved that the Faà di Bruno bialgebra is the reduced incidence bialgebra of the lattice of set partitions, and Joyal [**31**] observed that it can also be obtained directly (without a reduction step) from the category of surjections.

Brouder, Frabetti and Krattenthaler introduce the infinite series

$$A = \sum_{k=1}^{\infty} A_k/k! \in \mathbb{C}[[A_1, A_2, A_3, \ldots]],$$

check that the comultiplication extends to the power-series ring, and show that the Faà di Bruno formula can be formulated succinctly as

$$\Delta(A) = \sum_{k=1}^{\infty} A^k \otimes A_k/k! \tag{1}$$

(The exponent k is kth power in the ring.) The individual coefficients can be extracted from this formula.

This form of the Faà di Bruno formula is of importance in quantum field theory, where the role of the series A is played by the *connected Green function* [**5**], defined in the Connes–Kreimer Hopf algebra of Feynman graphs [**13**] as the sum of all connected graphs divided by their symmetry factors. Van Suijlekom [**47**] established a (multivariate) Faà di Bruno formula for the connected Green function, thereby vindicating the relevance of the Hopf algebra of graphs also in non-perturbative QFT. The coalgebraic Faà di Bruno formula (1) has also been exploited in the so-called exponential renormalisation [**17**]. Inspired by van Suijlekom's result, Gálvez, Kock and Tonks [**21**] proved a general Faà di Bruno formula in bialgebras of P-trees (P a finitary polynomial endofunctor), introducing categorical and homotopical methods which we further exploit in the present contribution to prove a Faà di Bruno formula in a much more general setting.

Beyond calculus, combinatorics, and classical applications to probability theory (for the latter, see for example [**32**]), Faà di Bruno-type formulae pop up in various contexts in the mathematical sciences. Most prominently perhaps in algebraic topology, in connection with complex cobordism [**41**] and vertex operator algebras [**43**], but also in areas such as control theory [**15, 27**], population genetics [**28**], and differential linear logic [**11**], to mention a few that have come to our attention. We do not know whether our contribution can be of any relevance in these contexts.

1. Outline of results and proof ingredients

1.1. **Heuristic outline.** Given an operad R (satisfying finiteness conditions, cf. 1.5 below), one can form its incidence bialgebra: as an algebra it is the polynomial ring in the set of iso-classes of operations of R. The monomials are interpreted as formal disjoint unions of operations. The comultiplication is the crucial structure: an operation is comultiplied by summing over all ways the operation can arise by operad substitution from a collection of operations fed into a single operation:

$$\Delta(r) = \sum_{r=b\circ(a_1,\ldots,a_n)} a_1 \cdots a_n \otimes b.$$

(The sum is over iso-classes of factorisations—to be technically correct it should rather be a homotopy sum, as will be detailed.) The comultiplication is extended multiplicatively to the whole polynomial ring, which therefore becomes a bialgebra, the *incidence bialgebra* of the operad [**25**].

Inside the completion of this bialgebra (the power series ring), we now define the *connected Green function* (by analogy with quantum field theory) to be the series consisting of all the operations themselves (but not their disjoint unions—this is the meaning of the word 'connected'), divided by their symmetry factors. This series G can be written as an infinite sum

$$G = \sum_n g_n$$

where g_n consists of all the operations of arity n (n a sequence of colours), divided by their symmetry factors. The comultiplication is shown to extend to the power series ring, and we can now state our general Faà di Bruno formula:

Main Theorem. (Cf. 5.6.)

$$\Delta(G) = \sum_k G^k \otimes g_k.$$

The two extreme examples of this construction are the following. When R is the terminal reduced operad Comm_+ (i.e. no nullary operations, and a single n-ary operation for each $n > 0$), then the formula is the classical (1) (cf. Example 7.1), with g_k denoting the unique operation in arity k (divided by $k!$), corresponding to $A_k/k!$. When R is the free operad on a finitary polynomial endofunctor P, then the operations are the P-trees, and g_n is given by iso-classes of P-trees with n leaves. The resulting Faà di Bruno formula in this case is that of Gálvez–Kock–Tonks [**21**] (cf. Example 7.4). More examples are given in Section 7.

1.2. **Formalisation.** It is likely that the formula could be proved by hand, just by expansion of series and brute computation. The difficulty in that approach is to handle correctly the symmetry factors that appear, as well illustrated by the computations of van Suijlekom [**46, 47**]. The insight of [**21**] in the case of trees was that the formula can be realised as the homotopy cardinality of an equivalence of groupoids (hence constituting a bijective proof), and that at the groupoid level all the symmetry factors take care of themselves automatically. The actual equivalence established in [**21**] involves groupoids of trees and trees with a cut, relying on specifics of the combinatorics of the Butcher–Connes–Kreimer bialgebra of trees.

The present contribution exploits the objective method initiated in [**21**], establishing the Faà di Bruno formula as the homotopy cardinality of an equivalence of groupoids, but takes a further abstraction step, which leads to a more general formula and a much simpler proof. We achieve this by leveraging some recent advances in category theory: on one hand, 2-categorical perspectives on operads and related structures discussed in [**51, 52, 53**], and on the other hand, the theory of decomposition spaces [**23, 24**]. With these tools, the proof of the equivalence of groupoids ends up being rather neat, emerging naturally from general principles.

1.3. **The objective method in a nutshell: groupoid slices instead of vector spaces.** It is well appreciated in combinatorics that bijective proofs represent deeper insight than formal algebraic ones. From the viewpoint of category theory, this deeper insight typically involves universal properties. Algebraic objects associated to combinatorial structures often have underlying vector spaces, generated freely by isomorphism classes of the combinatorial objects. It is a general hypothesis that whenever linear combinations involve symmetries—as exemplified in quantum field theory where formal sums of Feynman graphs are weighted by inverses of symmetry factors—they arise as homotopy cardinalities of groupoids rather than just cardinalities of sets; more precisely, as homotopy cardinalities of homotopy sums, as we recall below. Accordingly, algebraic identities in such situations should reflect equivalences of groupoids rather than bijections of sets.

A systematic way of expressing algebraic identities objectively is to replace vector spaces by slice categories. If B is a groupoid of combinatorial objects, then the basic vector space of interest is the free vector space on $\pi_0 B$, the set of iso-classes of objects in B. Just as a vector is a formal linear combination of elements in $\pi_0 B$ (i.e. a family of scalars indexed by $\pi_0 B$), an element $p : X \to B$ in the slice category $\boldsymbol{Grpd}_{/B}$ is a collection of groupoids indexed by B, the members of the family being the (homotopy) fibres X_b. Similarly, linear maps are represented by 'linear functors', in turn given by spans of groupoids. The ordinary linear algebra is obtained by taking the homotopy cardinality of the groupoid-level 'linear algebra' [**22**]. The homotopy cardinality of a family $p : X \to B$ in $\boldsymbol{Grpd}_{/B}$ is defined as

$$|p| = \sum_{b \in \pi_0 B} \frac{|X_b|}{|\mathrm{Aut}\, b|}\, \delta_b$$

which is an element in the vector space $\mathbb{Q}_{\pi_0 B}$ spanned by the symbols δ_b, one for each iso-class of objects in B.

1.4. **Brief explanation of the groupoid equivalence.** To explain the equivalence briefly, for simplicity we take the case where R is single-coloured, and gloss over a few technical details. The basic vector space is that spanned by the monomials in the (iso-classes of) operations of the operad R. Accordingly, we shall work with the groupoid $D_1 = \mathsf{S}C_1$, the free symmetric monoidal category on the groupoid C_1 of operations of R; more precisely, C_1 is the action groupoid (i.e. homotopy quotient) for the action of the symmetric groups on the sets of operations. So the basic slice is $\boldsymbol{Grpd}_{/D_1}$ whose cardinality is the vector space $\mathbb{Q}_{\pi_0 D_1}$ spanned by $\pi_0 D_1$. The *connected Green function* G is the formal series defined as the homotopy cardinality of the object $C_1 \to D_1$ in $\boldsymbol{Grpd}_{/D_1}$. The comultiplication map Δ is the cardinality of a certain span [**23**],

$$D_1 \xleftarrow{d_1} D_2 \xrightarrow{(d_2,d_0)} D_1 \times D_1$$

whose maps refer to a simplicial groupoid $D_\bullet : \triangle^{\mathrm{op}} \to \boldsymbol{Grpd}$ canonically constructed from the operad as a certain relative (two-sided) bar construction (cf. Section 3). The Faà di Bruno formula, which at the algebraic level is an equation in the vector space $\mathbb{Q}_{\pi_0 D_1} \otimes \mathbb{Q}_{\pi_0 D_1}$, is therefore supposed to be the cardinality of an equivalence of groupoids over $D_1 \times D_1$. Here is the equivalence (established in Proposition 3.11):

$$\begin{tikzcd}[column sep=tiny] C_1 \times_{D_1} D_2 \arrow[dr] & \simeq & \int^k (D_1)_k \times {}_k(C_1) \arrow[dl] \\ & D_1 \times D_1. & \end{tikzcd}$$

The left-hand side is precisely the definition of the comultiplication of the connected Green function $C_1 \to D_1$, so its homotopy cardinality is $\Delta(G)$. The right-hand side is a groupoid $D_1 \times_{D_0} C_1$, written as the homotopy sum of its homotopy fibres over D_0, the groupoid of finite sets and bijections, and it has to be unravelled a bit: $(D_1)_k$ denotes the k-component of the groupoid of families of operations, meaning those families that have k members. This is the same thing as k-tuples of operations, $(D_1)_k \simeq (C_1)^k$, whose homotopy cardinality is precisely G^k. The symbol ${}_k(C_1)$ denotes the homotopy fibre of C_1 over k under the arity map, so it is the groupoid of k-ary operations, and maps fixing the input slots. The integral sign designates a homotopy sum [**8, 22**],

$$\int^k (\ ) = \sum_k \frac{(\ )}{\operatorname{Aut} k}$$

where the division bar denotes homotopy quotient under the action of $\operatorname{Aut}(k)$, acting diagonally on $(D_1)_k \times {}_k(C_1)$. Since the action on the first factor is free, the action can be passed to the second factor, and the effect of it is to add morphisms to the groupoid ${}_k(C_1)$ to allow permutation of the inputs. Altogether, $\frac{{}_k(C_1)}{\operatorname{Aut}(k)}$ is the full groupoid of k-ary operations, and its homotopy cardinality is precisely g_k, showing that altogether the right-hand side of the equivalence has homotopy cardinality

$$\sum_k G^k \otimes g_k$$

as claimed.

The task is now to define the involved groupoids D_i and C_i correctly. It is a pleasing aspect of our approach that these groupoids come about by the standard general construction in algebraic topology known as the relative (two-sided) bar construction.

1.5. **Finiteness conditions.** At the groupoid level, in the 'objective' bialgebra $\boldsymbol{Grpd}_{/D_1}$, the Faà di Bruno formula holds for *any* operad. However, in order to be able to take cardinality, a certain finiteness condition must be imposed [**24**] (which is automatic in the two previously known cases, when either R is Comm_+ or free on a polynomial endofunctor). Namely, an operad R is called *locally finite* when for each operation r, the groupoid of possible decompositions $r = b \circ (a_1, \ldots, a_n)$ is homotopy finite. Equivalently, the map $d_1 : D_2 \to D_1$ is homotopy finite.

1.6. **Remark on grading and Hopf versus bialgebras.** The classical Faà di Bruno bialgebra is naturally graded, with $\deg A_k = k - 1$. Because of this minus-one, a shifted indexing convention is often used in the literature [**6, 16, 46, 47**], writing A_{k-1} instead of A_k. The present convention (also that of [**21**] and [**37**]) is dictated by the operadic approach, where it is essential that the superscript on G^k (counting k outputs) matches the subscript on g_k (counting k inputs).

It is also worth noting that it is essential to work with bialgebras rather than Hopf algebras. Our bialgebras are not connected, as in degree zero they are certain free symmetric monoidal categories, such as in the classical case the groupoid of finite sets and bijections (to which k belongs). Hopf algebras can be obtained by collapsing degree zero, but this amounts to throwing away the data controlling the match, as just described.

1.7. **Generalisations.** The arguments actually work the same for any situation $\mathsf{R} \Rightarrow \mathsf{S}$, of two polynomial monads, one cartesian over the other, but possibly over different slices. The construction also works for R-algebras rather than for R itself (which is actually the case of the terminal R-algebra). We describe these generalisations in Section 6.

In the operad case, S is the symmetric monoidal category monad. The comonoid structure comes from the combinatorics of R, and it will (almost) always be noncocommutative, as a consequence of the fact that operads have many inputs but only one output. The algebra structure, which is always free, is of a different nature, deriving from the fact that operads are considered internal to symmetric monoidal categories, by the choice of S. For general S, the outcome will not exactly be a bialgebra, but rather a free S-algebra in the category of coalgebras. Some care is needed to interpret the Faà di Bruno formula correctly in this more general setting, described in Section 6.

2. Operads and polynomial monads

Our main interest is in operads, but it is technically convenient to deal with them in the setting of polynomial monads, which also leads to a natural generalisation. The natural level of generality for our Faà di Bruno formula is that of one polynomial monad cartesian over another

$$\mathsf{R} \Rightarrow \mathsf{S}$$

in a double-category sense [**19**]. However, for expository reasons, and since it is the main case, we concentrate on the case of operads, namely when S is the symmetric monoidal category monad (2.5), which we assume in Sections 2–5. Then in Section 6 we explain how everything carries over readily to the general case.

2.1. **Groupoids and homotopy sums.** We freely use basic homotopy theory of groupoids, such as homotopy pullbacks and homotopy fibres, referring to [**8**] for all details. Here we content ourselves to briefly review the notion of homotopy sum, since it is a key point, accounting for the origin of the symmetry factors, as first exploited in [**21**].

It is plain that for a map of sets $E \to B$, the set E can be regarded as the sum of its fibres, $E \simeq \sum_{b\in B} E_b$. The same is true for groupoids, provided we use homotopy fibres and homotopy sums, as we now recall. Each homotopy fibre E_b comes with a canonical map to E, but it is not fully faithful. But the automorphism group $\mathrm{Aut}(b)$ acts on E_b canonically, and the action groupoid (also called homotopy quotient) $E_b/\mathrm{Aut}(b)$ does map to E fully faithfully; summing over one element b for each connected component in B then yields an equivalence of groupoids

$$\sum_{b\in\pi_0 B} \frac{E_b}{\mathrm{Aut}(b)} \simeq E.$$

The left-hand side is an example of a homotopy sum, and is denoted $\int^{b\in B} E_b$. It is an instance of a homotopy colimit, indexed by the groupoid B, just as an ordinary sum is a colimit indexed by a set. (The integral notation is standard, and is also compatible with general usage in category theory, since it is also an instance of a coend.) The great benefit of working with homotopy sums, is that they interact with homotopy pullbacks in the nicest way, precisely as ordinary sums interact with pullbacks in the category of sets.

The following lemma, which is a straightforward variation of this splitting into fibres, will be crucial for the Faà di Bruno formula.

Lemma 2.2. *Given a (homotopy) pullback square*

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

there is a natural equivalence of groupoids

$$P \simeq \int^{s\in S} X_s \times Y_s.$$

It is also an equivalence over $X \times Y$, meaning more precisely an equivalence in the weak slice $\boldsymbol{Grpd}_{/X \times Y}$.

2.3. **Polynomial monads, classically.** The theory of polynomial functors has roots in topology, representation theory, combinatorics, logic, and computer science. A standard reference is [**26**], which also contains pointers to those original developments.

A *polynomial* is a diagram of sets

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I'.$$

It defines a polynomial functor $\boldsymbol{Set}/I \to \boldsymbol{Set}/I'$ by the formula

$$t_! \circ p_* \circ s^*.$$

Polynomial functors form a double category in which the 2-cells are diagrams of the form

$$\begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l] \arrow[d] \arrow[r] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

Polynomial monads are horizontal monads in this double category, and the relevant monad maps are the vertical monad maps [**19**]. At the level of functors, this situation amounts to having one monad R on the slice category $\boldsymbol{Set}/I$, another monad S on the slice category $\boldsymbol{Set}/J$, and a map $F : I \to J$ for which $F_! : \boldsymbol{Set}/I \to \boldsymbol{Set}/J$ and its right adjoint F^* form a monad adjunction in the sense of [**52**]. This is turn amounts to having a natural transformation $\phi : F_!\mathsf{R} \to \mathsf{S}F_!$ making $(F_!, \phi)$ into a monad opfunctor in the sense of Street [**45**].

Polynomial monads over $\boldsymbol{Set}$ can account for nonsymmetric operads [**26**] and more generally sigma-free operads [**33**], but to account for general (symmetric) operads, at least groupoids are needed instead of sets.

2.4. **Polynomial functors over groupoids.** Polynomial functors over groupoids can be dealt with either in a homotopical setting as outlined in [**34**] or in a 2-categorical setting [**50**].

In the homotopical setting, the involved groupoids are only ever defined up to homotopy equivalence; one works with weak slices, and all notions are homotopy, e.g. homotopy pullbacks, homotopy fibres, etc., exploiting that in the homotopy sense groupoids form a locally cartesian closed category. Ultimately, the natural setting for this approach is that of ∞-groupoids, as adopted in [**23, 24**]. We follow 'tradition' in homotopy theory of denoting the weak slices $\boldsymbol{Grpd}_{/I}$, with a subscripted slash.

On the other hand, in the 2-categorical approach one works mostly with strict slices $\boldsymbol{Grpd}/I$ (for which we use non-subscripted slashes), strict pullbacks, and so on. (Actually the 2-categorical approach deals naturally with categories instead of groupoids, as exploited to good effect in [**50, 51, 52**], but for the present purposes we stick with groupoids.) The 2-category $\boldsymbol{Grpd}$ is not locally cartesian closed, so one has to assume the middle maps in the polynomial diagrams to be fibrations, in order for pullbacks to have right adjoints ('lowerstars'). Fibrations also play an important role to ensure that various 'strict' constructions involving pullbacks are homotopically meaningful. Such issues are handled efficiently in terms of the *fibration monads* [**44**], which turns general functors into fibrations. The weak slice $\boldsymbol{Grpd}_{/I}$ appears as the Kleisli category for the fibrations monad on the strict slice $\boldsymbol{Grpd}/I$, and the homotopy content is essentially controlled in terms of compatibilty with the fibration monads [**49, 51, 52**].

Both approaches will be exploited here: for the purpose of setting up the simplicial groupoid $D_\bullet$, we shall work 2-categorically, since it gives more precise results, and since a well-developed theory is already in place. But once that simplicial groupoid is in place, we shall pass to the homotopical setting, since the main object of interest is the weak slice $\boldsymbol{Grpd}_{/D_1}$.

2.5. **Key example: the symmetric monoidal category monad.** From now on, and until Section 6, $\mathsf{S} : \boldsymbol{Grpd} \to \boldsymbol{Grpd}$ denotes the symmetric monoidal category monad. It is polynomial, represented by the polynomial

$$1 \leftarrow \mathbb{B}' \to \mathbb{B} \to 1$$

where $\mathbb{B}$ is the groupoid of finite sets and bijections, and $\mathbb{B}'$ is the groupoid of finite pointed sets and basepoint-preserving bijections. The formula for evaluation is

$$X \mapsto \int^{n \in \mathbb{B}} \mathrm{Map}(\mathbb{B}'_n, X) \simeq \int^{n \in \mathbb{B}} X^{\underline{n}},$$

where $\underline{n}$ denotes the fibre over n. Note that $\mathbb{B}' \to \mathbb{B}$ is a discrete fibration (in fact the classifier for finite discrete maps of groupoids). Hence any monad cartesian over S will automatically be finitary again, meaning that all operations have finite discrete arity.

2.6. **Operads as polynomial monads.** By operad we mean coloured symmetric operad in the category of sets. It was shown in [**51**] (Theorem 3.3) that operads are the same thing as polynomial monads cartesian over the symmetric monoidal category monad

$$\begin{tikzcd} I \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[r] \arrow[d] & I \arrow[d] \\ 1 & \mathbb{B}' \arrow[l] \arrow[r] & \mathbb{B} \arrow[r] & 1 \end{tikzcd}$$

for which I is a set, and $B \to \mathbb{B}$ is a discrete fibration. The polynomial viewpoint on operads has proven very useful in homotopy theory [**4**].

The equivalence goes as follows. Given an operad R, let I be its set of colours. Let B be the action groupoid (i.e. homotopy quotient) of the action of the symmetric groups on the sets of operations of each arity. More precisely, for the symmetric-group action $\mathsf{R}_n \times \mathfrak{S}_n \to \mathsf{R}_n$, the action groupoid has as objects the elements in R_n, and an arrow from r to r' for each $g \in \mathfrak{S}_n$ such that $r.g = r'$. Finally, B is the disjoint union of these action groupoids, with its canonical map to $\mathbb{B}$, itself the disjoint union of the classifying spaces of the $\mathfrak{S}_n$. This is a discrete fibration, whose fibre over n is the set R_n, by the standard fibre sequence for action groupoids [**8**]. Note that the action respects colours; for this reason it is meaningful to define the map $B \to I$ by assigning to an operation its output colour. The groupoid E consists of operations with a marked input slot. The map $E \to I$ assigns to a marked operation the colour of its marked input. The fibre of $E \to B$ over an operation r is the set of its input slots. The monad structure on the polynomial endofunctor comes precisely from the substitution operation of the operad R. The fact that this monad on $\boldsymbol{Grpd}/I$ is itself polynomial, implies that it is cartesian.

Conversely, given a polynomial monad cartesian over S as above, the discrete fibration $B \to \mathbb{B}$ induces a $\mathfrak{S}$-set which is the set of operations. The set of n-ary operations is the (homotopy) fibre of $B \to \mathbb{B}$ over n. The fact that there is a pullback square gives the set of input slots a linear order (interpreting $\mathbb{B}$ as the groupoid of linear orders $\underline{n} = \{1, 2, \ldots, n\}$ and not-necessarily-monotone maps), as befits an operad in the classical sense. The operad substitution comes from the monad multiplication.

Although not central to this article, the monads R, S and the fibrations monads described above, are all 2-*monads*. In particular since R is a 2-monad, one may consider its strict algebras, and from [**51**] these were understood to be 'weakly-equivariant' $\boldsymbol{Cat}$-valued algebras of the operad R. The

2-dimensionality of R and S enables the definition of an 'algebra of R internal to an algebra of S' to be made, as in [**3, 4, 52**], and it is these that correspond to the algebras of the operad R in the usual sense. From now on, the symbol R refers rather to the polynomial monad, which is what we actually work with.

3. Monad adjunctions and the relative (2-sided) bar construction

3.1. **Set-up.** We consider the situation as in 2.6, where we have a polynomial monad R on $\boldsymbol{Grpd}/I$, cartesian over the symmetric monoidal category monad S on $\boldsymbol{Grpd}$, represented altogether by the polynomial diagram

$$\begin{tikzcd} I \arrow[d, "F"'] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[r] \arrow[d] & I \arrow[d, "F"] \\ 1 & \mathbb{B}' \arrow[l] \arrow[r] & \mathbb{B} \arrow[r] & 1 \end{tikzcd}$$

The two monads R and S are intertwined by means of the functor $F_!$ and the natural transformation

$$\phi : F_!\mathsf{R} \Rightarrow \mathsf{S}F_!$$

forming together a monad opfunctor in the sense of Street [**45**], a monad adjunction in the sense of [**52**], or a vertical morphism of horizontal monads in the double-category setting [**19**].

It is a general fact that the natural transformation ϕ is cartesian. This follows because its ingredients are the unit and counit of lowershriek-upperstar adjunctions and an instance of the Beck–Chevalley isomorphism. See [**52**, § 3.3] for details.

3.2. **A simplicial-object-with-missing-top-face-maps.** Let A denote the terminal object in $\boldsymbol{Grpd}/I$, and let $\alpha : \mathsf{R}A \to A$ denote the unique map from $\mathsf{R}A$ in $\boldsymbol{Grpd}/I$. (Note that α has underlying map of groupoids $B \to I$.) The pair (A, α) is the terminal R-algebra.

In $\boldsymbol{Grpd}/I$ there is induced a natural simplicial-object-with-missing-top-face-maps

$$\begin{tikzcd} A \arrow[r, shift left, "s_0"] & \mathsf{R}A \arrow[l, shift left, "d_0"] \arrow[r, shift left=3, "s_1"] \arrow[r, shift right, "s_0"] & \mathsf{RR}A \arrow[l, shift left, "d_1"] \arrow[l, shift left=-3, "d_0"] \arrow[r, shift left=5, "s_2"] \arrow[r, shift left=1, "s_1"] \arrow[r, shift right=3, "s_0"] & \mathsf{RRR}A \arrow[l, shift left=3, "d_2"] \arrow[l, shift right=1, "d_1"] \arrow[l, shift right=5, "d_0"] & \cdots \end{tikzcd} \tag{2}$$

The bottom face maps come from the action $\alpha : \mathsf{R}A \to A$, and the remaining face and degeneracy maps come from the monad structure. Hence the basic maps are

$$\begin{tikzcd} A \arrow[r, shift left, "\eta_A"] & \mathsf{R}A \arrow[l, shift left, "\alpha"] \arrow[r, shift left=3, "\eta_{\mathsf{R}A}"] \arrow[r, shift right, "\mathsf{R}\eta_A"] & \mathsf{RR}A \arrow[l, shift left, "\mu_A"] \arrow[l, shift left=-3, "\mathsf{R}\alpha"] \end{tikzcd}$$

and in general, $s_k : \mathsf{R}^n A \to \mathsf{R}^{n+1}A$ is given by $\mathsf{R}^{n-k}\eta_{\mathsf{R}^k A}$ and $d_k : \mathsf{R}^{n+1}A \to \mathsf{R}^n A$ is given by $\mathsf{R}^{n-k}\mu_{\mathsf{R}^{k-1}A}$, with the convention that $\mu_{\mathsf{R}^{-1}A} = \alpha$.

We now apply $F_!$ to the diagram (2) above, to obtain inside $\boldsymbol{Grpd}$ a simplicial-object-with-missing-top-face-maps which we denote $C_\bullet$:

$$C_\bullet : \quad \begin{tikzcd} F_!A \arrow[r, shift left, "s_0"] & F_!\mathsf{R}A \arrow[l, shift left, "d_0"] \arrow[r, shift left=3, "s_1"] \arrow[r, shift right, "s_0"] & F_!\mathsf{RR}A \arrow[l, shift left, "d_1"] \arrow[l, shift left=-3, "d_0"] \arrow[r, shift left=5, "s_2"] \arrow[r, shift left=1, "s_1"] \arrow[r, shift right=3, "s_0"] & F_!\mathsf{RRR}A \arrow[l, shift left=3, "d_2"] \arrow[l, shift right=1, "d_1"] \arrow[l, shift right=5, "d_0"] & \cdots \end{tikzcd}$$

Finally we apply S: the diagram now acquires the missing top face maps, and constitutes altogether a simplicial object in $\boldsymbol{Grpd}$ ([**52**], Lemma 4.3.2) which we denote $D_\bullet$:

$$D_\bullet : \quad \begin{tikzcd} \mathsf{S}F_!A \arrow[r, shift left, "s_0"] & \mathsf{S}F_!\mathsf{R}A \arrow[l, dotted, shift left=5, "d_1"'] \arrow[l, shift left, "d_0"] \arrow[r, shift left=3, "s_1"] \arrow[r, shift right, "s_0"] & \mathsf{S}F_!\mathsf{RR}A \arrow[l, dotted, shift left=7, "d_2"'] \arrow[l, shift left, "d_1"] \arrow[l, shift left=-3, "d_0"] \arrow[r, shift left=5, "s_2"] \arrow[r, shift left=1, "s_1"] \arrow[r, shift right=3, "s_0"] & \mathsf{S}F_!\mathsf{RRR}A \arrow[l, dotted, shift left=9, "d_3"'] \arrow[l, shift left=3, "d_2"] \arrow[l, shift right=1, "d_1"] \arrow[l, shift right=5, "d_0"] & \cdots \end{tikzcd}$$

The new top face maps are given by

$$d_\top := \mu^{\mathsf{S}} \circ \mathsf{S}(\phi).$$

For example, the first of the top face maps is given as

$$\begin{tikzcd}
& \mathsf{SS}F_!A \arrow[dl, "\mu^S"'] & \\
\mathsf{S}F_!A & & \mathsf{S}F_!\mathsf{R}A. \arrow[ul, "\mathsf{S}\phi"'] \arrow[ll, "d_1"]
\end{tikzcd}$$

Proposition 3.3 ([**52**], Proposition 4.4.1)**.** *The simplicial object $D_\bullet$ is a strict category object.*

The statement is that all the squares

$$\begin{tikzcd}
D_{n+2} \arrow[r, "d_0"] \arrow[d, "d_{n+2}"'] & D_{n+1} \arrow[d, "d_{n+1}"] \\
D_{n+1} \arrow[r, "d_0"'] & D_n
\end{tikzcd}$$

are strict pullbacks. As an illustration, the first of these squares ($n = 0$) is the following, by unravelling the definition of the top face maps:

$$\begin{tikzcd}
\mathsf{S}F_!\mathsf{RR}A \arrow[r, "\mathsf{S}F_!\mathsf{R}\alpha"] \arrow[d, "\mathsf{S}(\phi_{\mathsf{R}A})"'] & \mathsf{S}F_!\mathsf{R}A \arrow[d, "\mathsf{S}(\phi_A)"] \\
\mathsf{SS}F_!\mathsf{R}A \arrow[r, "\mathsf{SS}F_!\alpha"] \arrow[d, "\mu^{\mathsf{S}}_{F_!\mathsf{R}A}"'] & \mathsf{SS}F_!A \arrow[d, "\mu^{\mathsf{S}}_{F_!A}"] \\
\mathsf{S}F_!\mathsf{R}A \arrow[r, "\mathsf{S}F_!\alpha"'] & \mathsf{S}F_!A.
\end{tikzcd} \tag{3}$$

The bottom square is a naturality square for the monad multiplication, and is therefore a strict pullback. The top square is S applied to a naturality square for ϕ. Since ϕ is cartesian and S preserves pullbacks, this is again cartesian.

3.4. **Remark.** This simplicial object is a variation of the 2-sided bar construction, which has a long history in algebraic topology [**40**]. The construction here from an operad is important in category theory since its codescent object (its lax colimit) is the classifier for R-algebras (at the present level of generality in symmetric monoidal categories), i.e. the universal symmetric monoidal category containing an internal R-algebra, cf. [**4**], [**52**]. In the present work, we do not take the codescent object, but work directly with the simplicial groupoid.

Proposition 3.5. *The simplicial groupoid $D_\bullet$ is a Segal groupoid.*

Proof. Since we already know that it is a category object, the statement is that the strict pullback squares in question are also homotopy pullbacks, a typical result for the 2-categorical approach to polynomial functors. Since the Segal squares are composed of components of ϕ and components of μ^{S} (as in diagram (3)), it is enough to show that $\mathsf{S} \circ F_!$ (the codomain of ϕ) and S itself (the codomain of μ^{S}) satisfy the conditions of the next lemma, which is clear since I is discrete and $\mathbb{B}' \to \mathbb{B}$ is a split opfibration. □

Lemma 3.6. *A strictly cartesian 2-natural transformation between polynomial 2-functors (between strict slices of* ***Grpd****) is also homotopy cartesian provided the codomain 2-functor* $\mathbf{Grpd}/I \to \mathbf{Grpd}/J$ *has its lowerstar component along a split opfibration, and the groupoid* I *is discrete.*

Proof. Proposition 4.6.5 of [**53**] states that a strictly cartesian 2-natural transformation between polynomial 2-functors (between strict slices of ***Grpd***) is also homotopy cartesian if just the codomain 2-functor is *familial* [**49**], which essentially means that it preserves fibrations. On the other hand, familiality is implied by the following slight variation of Theorem 4.4.5 of [**50**], whose notation we use freely: in the proof given in [**50**], the condition that the lowershriek component is a fibration is not needed. For the required factorisation of $\mathcal{K}/I \to \mathcal{K}/B$ through $U^{\Phi_{\mathcal{K},B}}$ one only needs s^* and p_* to lift to the level of fibrations, the rest of the proof goes as in [**50**]. ☐

Proposition 3.7. *The category object* $D_\bullet := \mathsf{S}F_!\mathsf{R}^\bullet A$ *is a symmetric monoidal category object in* ***Grpd****, and its structure maps* $\otimes : \mathsf{S}D_\bullet \to D_\bullet$ *are cartesian and homotopy cartesian on degeneracy maps and inner face maps.*

The symmetric monoidal structure just comes from the fact that levelwise $D_\bullet$ is S of something, hence is in fact a *free* S-algebra. The cartesianness is just the cartesianness of the structure maps of the monad S—this applies to all but the top face maps. Homotopy cartesianness follows from Lemma 3.6.

3.8. **The connected Green function.** The (weak) slice $\boldsymbol{Grpd}_{/D_1}$ contains a canonical element, namely

$$G : C_1 \to D_1$$

which we call the *connected Green function*. Recalling that $C_1 = F_!\mathsf{R}A$ and that $D_1 = \mathsf{S}F_!\mathsf{R}A$, the connected Green function is simply $\eta^{\mathsf{S}}_{F_!\mathsf{R}A}$, the unit for the monad S, the inclusion of singleton families into all families.

We shall also need to split G into its fibres over D_0. Denote by w an object in $D_0 = \mathsf{S}F_!A$, a tuple of objects in $F_!A$. Write ${}_w(D_1)$ for the homotopy fibre over $w \in D_0$ of the face map $d_1 : D_1 \to D_0$ (reserving the notation $(D_1)_w$ for the (homotopy) fibre over $w \in D_0$ of the other face map $d_0 : D_1 \to D_0$). For each $w \in D_0$, consider the (homotopy) pullback squares

$$\begin{tikzcd}
{}_w(C_1) \arrow[r] \arrow[d, "G_w"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \arrow[d, "G"] \\
{}_w(D_1) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & D_1 \arrow[d, "d_1"] \\
1 \arrow[r, "\ulcorner w \urcorner"'] & D_0
\end{tikzcd}$$

so that we have (as in 2.1)

$$G = \int^w G_w.$$

G_w is thus the w-ary part of the connected Green function.

Lemma 3.9. *We have (homotopy) pullback squares*

$$\begin{tikzcd} C_1 \arrow[d] & C_2 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dr, phantom, "\lrcorner"] & C_1 \arrow[d] \\ D_1 & D_2 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, "d_2"'] \arrow[dr, phantom, "\lrcorner"] & D_1 \arrow[d, "d_1"] \\ & D_1 \arrow[r, "d_0"'] & D_0 \end{tikzcd}$$

Proof. The top squares are cartesian since they are naturality squares for $\eta : \mathrm{Id} \Rightarrow \mathsf{S}$, and homotopy cartesian by Lemma 3.6. For the bottom square, the assertion follows from 3.5. □

Stacking the right-hand squares we get the following result, which is an abstraction of the Key Lemma 5.5 of [**21**].

Corollary 3.10. *We have a canonical equivalence of groupoids*

$$C_2 \simeq D_1 \times_{D_0} C_1.$$

This pullback we now split into its fibres over D_0 as in Lemma 2.2:

Proposition 3.11. *We have a canonical equivalence*

$$C_2 \simeq \int^w (D_1)_w \times {}_w(C_1).$$

This equivalence is over $D_1 \times C_1$ and hence also over $D_1 \times D_1$.

This is the essence of the Faà di Bruno formula, as we proceed to explain in the following sections. The groupoid C_2 (with its map to $D_1 \times D_1$) is the left-hand side of the Faà di Bruno formula, and $\int^w (D_1)_w \times {}_w(C_1)$ (with its map to $D_1 \times D_1$) is the right-hand side.

We now analyse the groupoid $(D_1)_w$, for $w \in D_0 = \mathsf{S}C_0$. Recall that S is defined by the polynomial $1 \leftarrow \mathbb{B}' \to \mathbb{B} \to 1$. For $n \in \mathbb{B}$, let $\underline{n}$ denote the fibre over n of the projection $\mathbb{B}' \to \mathbb{B}$. With this notation, the formula for evaluation of S reads $\mathsf{S}(X) = \int^{n \in \mathbb{B}} \mathrm{Map}(\underline{n}, X)$. The element $w \in D_0 = \mathsf{S}C_0 = \int^{n \in \mathbb{B}} \mathrm{Map}(\underline{n}, C_0)$ thus amounts to a map $w : \underline{n} \to C_0$ (for some n).

Lemma 3.12. *For fixed $w : \underline{n} \to C_0$, we have a canonical equivalence*

$$(D_1)_w \simeq \mathrm{Map}_{/C_0}(\underline{n}, C_1) =: C_1^w.$$

Here $\underline{n}$ is considered over C_0 via w, and C_1 is considered over C_0 via d_0.

Proof. The map $d_0 : D_1 \to D_0$ is S applied to $d_0 : C_1 \to C_0$, and hence, by definition of S, can be written

$$\int^{n \in \mathbb{B}} \big(\mathrm{Map}(\underline{n}, C_1) \to \mathrm{Map}(\underline{n}, C_0) \big)$$

which for fixed n is just post-composition with $d_0 : C_1 \to C_0$. The w-fibre of $D_1 \to D_0$ is therefore computed by the standard slice-mapping-space fibre sequence (see [**8**]):

$$\begin{tikzcd} \mathrm{Map}_{/C_0}(w, d_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\underline{n}, C_1) \arrow[d, "\text{post } d_0"] \\ 1 \arrow[r, "\ulcorner w \urcorner"'] & \mathrm{Map}(\underline{n}, C_0). \end{tikzcd}$$

□

This lemma is an abstraction of Lemma 5.2 in [**21**]. It can be interpreted as saying that an object of $(D_1)_w$ is an n-tuple of elements in C_1, with specified output colours given by the n-tuple $w : \underline{n} \to C_0$. In the lemma we have already introduced the notation C_1^w to reflect this interpretation.

The interpretation so far is phrased abstractly in terms of C_i, already with a view towards the generalisations in Section 6. In fact we can unpack further, to get down to groupoids of operations of the operad R. As a first step, we unravel the square in Corollary 3.10:

Lemma 3.13. *The pullback square*

$$\begin{tikzcd} C_2 \arrow[r, "d_0"] \arrow[d, "d_2 \circ \eta"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_2 \circ \eta"] \\ D_1 \arrow[r, "d_0"'] & D_0 \end{tikzcd}$$

is naturally identified with the naturality square for ϕ,

$$\begin{tikzcd} F_! \mathsf{R}\mathsf{R}A \arrow[r, "F_! \mathsf{R}\alpha"] \arrow[d, "\phi_{\mathsf{R}A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_! \mathsf{R}A \arrow[d, "\phi_A"] \\ \mathsf{S}F_! \mathsf{R}A \arrow[r, "\mathsf{S}F_! \alpha"'] & \mathsf{S}F_! A. \end{tikzcd} \tag{4}$$

Proof. The square is composed of two squares as in Lemma 3.9, the bottom being in turn composed of the two squares in (3). Altogether three squares are stacked, with sides η, then $\mathsf{S}\phi$, then μ. Now η and $\mathsf{S}\phi$ can be interchanged by naturality, and then η and μ cancel out, by the unit law of S, leaving only the asserted square (4), naturality for ϕ. $\square$

Now we unpack further. First note that $F_! A = I$ and $F_! \mathsf{R}A = B$, with reference to the polynomial $I \leftarrow E \to B \to I$ representing R. The effect on objects of $\phi_A : B \to \mathsf{S}I$ is to send an operation $b : (i_1, ..., i_n) \to i$ to the input sequence $(i_1, ..., i_n)$. The effect on objects of $\mathsf{S}F_! \alpha : \mathsf{S}B \to \mathsf{S}I$ is to send a sequence of operations $(b_1, ..., b_n)$ to the sequence of output colours of the b_j. For an arbitrary object $w = (i_1, ..., i_n)$ of $\mathsf{S}I$, the homotopy fibres ${}_wB$ and $(\mathsf{S}B)_w$ of these functors are easily computed. The groupoid ${}_wB$ has

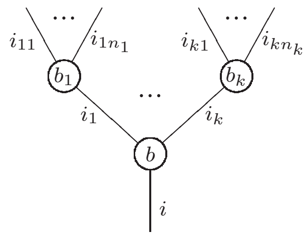
- objects triples (ρ, b, j) where $\rho \in \Sigma_n$ and $b : (i_{\rho 1}, ..., i_{\rho n}) \to j$ is an operation of R.
- arrows $(\rho, b, j) \to (\rho', b', j')$ are $\psi \in \Sigma_n$ such that $\rho = \rho' \psi$ and $b = b' \psi$.

Thus ${}_wB$ is a groupoid of operations with input sequence some permutation of $(i_1, ..., i_n)$. The groupoid $(\mathsf{S}B)_w$ has

- objects pairs $(\rho, (b_1, ..., b_n))$ where $\rho \in \Sigma_n$ and $b_j \in B$ such that $tb_j = i_{\rho j}$ for $1 \leqslant j \leqslant n$.
- arrows $(\rho, (b_1, ..., b_n)) \to (\rho', (b'_1, ..., b'_n))$ are permutations $\psi \in \Sigma_n$ and $(\psi_1, ..., \psi_n)$, such that $\rho = \rho' \psi$ and $b_j = b'_{\psi(j)} \psi_j$.

Thus $(\mathsf{S}B)_w$ is a groupoid of n-tuples of operations with bijection associating output colours with $(i_1, ..., i_n)$. As in Lemma 3.12, it is natural to denote $(\mathsf{S}B)_w$ as B^w. The groupoid $C_2 = F_! \mathsf{R}\mathsf{R}A$ is the groupoid of operations of R labelled by operations of R, described explicitly in the proof of

Proposition 3.6 of [**51**]. In pictorial terms, an object here is depicted as



Thus Proposition 3.11 unpacks to

Corollary 3.14. *For any operad* R*, with notation as above,*

$$C_2 \simeq \int^{w \in \mathsf{S}I} B^w \times {}_w B$$

over $\mathsf{S}B \times B$*.*

in clear analogy with the classical Faà di Bruno formula, especially in the uncoloured case, $I = 1$, where it reduces to $C_2 \simeq \int^n B^n \times {}_n B$.

4. Monoidal decomposition spaces and bialgebras

We now proceed to explain the bialgebra structure (at the groupoid level) in which to interpret the equivalence established above. The actual work was done above; we just need to apply the results of Gálvez–Kock–Tonks [**23**], and our task is only to explain how it works.

4.1. **Decomposition spaces and coalgebras.** The notion of decomposition space was introduced in [**23**], as a generalisation of posets and Möbius categories for the purpose of defining incidence coalgebras. A decomposition space is a simplicial groupoid $X_\bullet : \triangle^{\mathrm{op}} \to \boldsymbol{Grpd}$ (in full generality a simplicial ∞-groupoid) satisfying an exactness condition ensuring that the canonical span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1 \tag{5}$$

induces a homotopy-coherently coassociative coalgebra structure on $\boldsymbol{Grpd}_{/X_1}$ in a sense we shall now detail. The precise exactness condition is not necessary in the present work: all we need to know is that Segal spaces are decomposition spaces [**23**, Proposition 3.5].

The idea behind the coalgebra construction is simple, and goes back to Leroux [**39**]: an arrow f of a category object is comultiplied by the formula

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b.$$

The sum is over all ways of factoring f into two arrows, generalising the way intervals are comultiplied in the classical incidence coalgebra of a poset [**30**]. The above span (5) defines a linear functor

$$\boldsymbol{Grpd}_{/X_1} \xrightarrow{(d_2,d_0)_! \circ d_1^*} \boldsymbol{Grpd}_{/X_1 \times X_1}$$

which is precisely the objective version of Δ: pullback to X_2 means taking all triangles with long edge f, and then the lowershriek means returning the two short edges. (*Linear functor* means that it preserves homotopy sums; the linear functors are precisely those given by pull-push along spans [**22**].)

Similarly, the span

$$X_1 \xleftarrow{s_0} X_0 \xrightarrow{u} 1$$

defines the linear functor

$$\boldsymbol{Grpd}_{/X_1} \xrightarrow{u_! \circ s_0^*} \boldsymbol{Grpd}$$

which is the counit for Δ.

Altogether, $\boldsymbol{Grpd}_{/X_1}$ becomes a coalgebra object in the symmetric monoidal 2-category $(\boldsymbol{LIN}, \otimes, \boldsymbol{Grpd})$ [**22**], whose objects are (weak) slices and whose morphisms are linear functors. The monoidal structure $\otimes$ is given by the formula

$$\boldsymbol{Grpd}_{/A} \otimes \boldsymbol{Grpd}_{/B} = \boldsymbol{Grpd}_{/A\times B}$$

with neutral object $\boldsymbol{Grpd}_{/1}$, and the symmetry is induced by the canonical symmetry of the cartesian product.

4.2. **Monoidal decomposition spaces.** In general, a sufficient condition for a simplicial map $f : Y_\bullet \to X_\bullet$ between decomposition spaces to induce a coalgebra homomorphism on incidence coalgebras, is that f be *CULF* [**23**], which stands for *conservative* (meaning that it does not invert any arrows) and *unique lifting of factorisations* (meaning that for an arrow $a \in Y_1$, there is a one-to-one correspondence between the factorisations of a in $Y_\bullet$ and the factorisations of $f(a)$ in $X_\bullet$). A simplicial map is CULF precisely when it is homotopy cartesian on degeneracy maps and inner face maps [**23**].

Recall that a bialgebra is a monoid object in the category of coalgebras, meaning that multiplication and unit are homomorphisms of coalgebras. Accordingly, in order to induce a bialgebra structure on $\boldsymbol{Grpd}_{/X_1}$, we need on $X_\bullet$ a monoidal structure which is CULF. This motivates defining a *monoidal decomposition space* [**23**] to be a decomposition space $X_\bullet$ equipped with a monoidal structure $\otimes : \mathsf{S}X_\bullet \to X_\bullet$, whose structure maps are CULF.

4.3. **Bialgebra structure on $\boldsymbol{Grpd}_{/D_1}$.** Coming back now to the simplicial groupoid $D_\bullet = \mathsf{S}F_!\mathsf{R}^\bullet A$, we have precisely such a symmetric monoidal structure, namely given by the multiplication map $\mu^{\mathsf{S}} : \mathsf{S}(\mathsf{S}F_!\mathsf{R}^\bullet A) \to \mathsf{S}F_!\mathsf{R}^\bullet A$, which is CULF by Proposition 3.7. In conclusion, by Theorem 7.3 and Proposition 9.5 of [**23**], we have:

Proposition 4.4. *The slice category $\boldsymbol{Grpd}_{/D_1}$ is a bialgebra object in $(\boldsymbol{LIN}, \otimes, \boldsymbol{Grpd})$.*

With the current choice of S it is even a 'symmetric' bialgebra, meaning that the homotopy cardinality will be a commutative bialgebra, as we shall see in the next section.

We can now give the bialgebra reformulation of the equivalence in Proposition 3.11, which is the Faà di Bruno formula at the objective level:

4.5. **Theorem.** *We have the following equivalence in $\boldsymbol{Grpd}_{/D_1\times D_1}$:*

$$\Delta(G) = \int^w G^w \times G_w.$$

5. Finiteness conditions and homotopy cardinality

When working at the objective level of groupoid slices, the results so far hold for *any* operad R. However, in order to take cardinality, it is necessary to subject R to certain finiteness conditions [**24**].

In this section we explain the procedure of taking homotopy cardinality, following [**22**]. In that paper, the setting is that of ∞-groupoids, but everything works also for 1-groupoids [**8**].

5.1. **Homotopy cardinality of groupoids and slices [22].** A groupoid X is called *homotopy finite* when it has only finitely many connected components, and all its automorphism groups are finite. In that case, the homotopy cardinality of X is defined as

$$|X| := \sum_{x \in \pi_0 X} \frac{1}{|\mathrm{Aut}\, x|}.$$

Let $\boldsymbol{grpd}$ (lowercase) denote the category of finite groupoids.

Groupoids of combinatorial objects are *not* usually finite, because they have infinitely many components, but they usually have finite automorphism groups. Such groupoids are called *locally finite* [**22**]. From now on we assume that all groupoids are locally finite. For X a locally finite groupoid, the basic vector space is $\mathbb{Q}_{\pi_0 X}$, the free vector space on the set of iso-classes of objects in X. We denote the basis elements δ_x (for $x \in \pi_0 X$). Since linear combinations in a vector space are finite by definition, the correct groupoid slice to consider is $\boldsymbol{grpd}_{/X}$ of finite groupoids over X. The cardinality of an element $A \to X$ in $\boldsymbol{grpd}_{/X}$ is the vector

$$\sum_{x \in \pi_0 X} \frac{|A_x|}{|\mathrm{Aut}\, x|}\, \delta_x \quad \in \mathbb{Q}_{\pi_0 X}.$$

Linear maps $\mathbb{Q}_{\pi_0 X} \to \mathbb{Q}_{\pi_0 Y}$ are modelled by linear functors $\boldsymbol{grpd}_{/X} \to \boldsymbol{grpd}_{/Y}$, in turn given by spans of *finite type*

$$X \xleftarrow{p} M \xrightarrow{q} Y,$$

meaning that p is a finite map (i.e. has homotopy-finite homotopy fibres).

The notion of homotopy cardinality has all the expected properties: it takes equivalences to equalities, it preserves products, sums, quotients—and hence homotopy sums; it takes monoidal structures to monoids.

5.2. **Locally finite decomposition spaces.** For $X_\bullet$ a decomposition space, we consider the finite-groupoid slice $\boldsymbol{grpd}_{/X_1} \subset \boldsymbol{Grpd}_{/X_1}$, which is well behaved assuming X_1 is locally finite. For the coalgebra structure maps $\Delta : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}_{/X_1 \times X_1}$ and $\varepsilon : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$ to descend to the finite slice $\boldsymbol{grpd}_{/X_1}$ it is therefore sufficient to require that $d_1 : X_2 \to X_1$ and $s_0 : X_0 \to X_1$ be finite maps. Decomposition spaces with this property (and X_1 locally finite) are called *locally finite* [**24**, §7] (extending the notion for posets [**30**]). (It may be noted that there are no conditions on the algebra structure: it always descends to finite-groupoid slices, since it is a pure lowershriek operation.)

5.3. **Locally finite operads (and monads).** Coming back to operads, we can ensure the local finiteness condition on $D_\bullet$ by the following requirement. Call a monad R *locally finite* when $\mu : \mathsf{RR} \Rightarrow \mathsf{R}$ and $\eta : \mathrm{Id} \Rightarrow \mathsf{R}$ are finite natural transformations (i.e. all components are finite maps). What it amounts to is that for every operation r, there are only finitely many ways of writing it

$$r = b \circ (a_1, \ldots, a_n)$$

for b an n-ary operation (and a_i operations whose arities add up to that of r).

As an non-example, the commutative monoid monad S is *not* locally finite, because the identity operation $u \in \mathsf{S}_1$ could be obtained in infinitely many ways as a composition: by filling $n-1$ nullary operations into an n-ary operation (for all n). In contrast, cf. Example 7.1 below, the commutative semimonoid monad S_+ *is* locally finite, since there are no nullary operations to screw things up. Let us observe that the existence of nullary operations is not formally an obstruction to being locally finite, as long as they are not subject to relations. For example, every free monad (on a polynomial endofunctor) is locally finite, cf. Example 7.4 below.

The local finiteness of R makes makes $C_\bullet$ locally finite (it is not quite a simplicial object, because the top face maps are missing, but its does feature the face and degeneracy maps on which the condition is measured). Since $F_!$ and S preserve finite maps, $D_\bullet$ is also locally finite.

5.4. **The connected Green function as a homotopy cardinality.** The connected Green function lives in the completion of $\mathbb{Q}_{\pi_0 D_1}$, where we allow infinite sums (but not infinite coefficients). This (profinite-dimensional) vector space arises as the homotopy cardinality of the bigger slice $\boldsymbol{Grpd}^{\text{rel.fin.}}_{/D_1}$ whose objects are finite maps $E \to D_1$ (but E itself not required finite). The relevant notion of linear map (continuous in a certain 'pro' sense) are given by spans

$$X \xleftarrow{p} M \xrightarrow{q} Y$$

where instead the right leg q is finite [**22**].

In order to formulate the Main Theorem at the vector space level, we have to justify that the comultiplication extends to this bigger slice, which is to say we must check that $D_1 \times_{D_0} D_1 \to D_1 \times D_1$ is a finite map. This is a standard argument [**22**]: this map sits naturally in a homotopy pullback square

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_0 \arrow[d, "\text{diag.}"] \\ D_1 \times D_1 \arrow[r] & D_0 \times D_0 \end{tikzcd}$$

so it is enough to show that the diagonal $D_0 \to D_0 \times D_0$ is finite. It is automatically discrete (i.e. 0-truncated) since D_0 is 1-truncated, and the homotopy fibre at a point (w, w) is essentially the set of automorphisms of w, which is finite (for all w) if and only if D_0 is locally finite. Since we have already assumed that D_1 is locally finite, and that $s_0 : D_0 \to D_1$ is finite, it follows that D_0 is again locally finite. Therefore, the comultiplication extends to the bigger slice $\boldsymbol{Grpd}^{\text{rel.fin.}}_{/D_1}$ as required. It may be noted that the counit does *not* extend to $\boldsymbol{Grpd}^{\text{rel.fin.}}_{/D_1}$. This would require $D_0 \to 1$ to be a finite map, which is never the case since it is S of something. This means that the coalgebra $\boldsymbol{Grpd}^{\text{rel.fin.}}_{/D_1}$ is not counital. This is not really an issue: our main interest is the counital coalgebra $\boldsymbol{grpd}_{/D_1}$. The extension to $\boldsymbol{Grpd}^{\text{rel.fin.}}_{/D_1}$ is introduced only to be able to state the Faà di Bruno formula efficiently in terms of the connected Green function—it does not involve the counit at all.

5.5. **Cardinality of the main equivalence.** We assume now that R is locally finite, and proceed to take homotopy cardinality of the main equivalence. For clarity, we temporarily use underline notation for the symbols at the algebra level. By construction, the family $G : C_1 \to D_1$ has cardinality $\underline{G}$, and the linear functor Δ has cardinality $\underline{\Delta}$. Hence the left-hand side of the equation is clear: the cardinality of C_2 is $\underline{\Delta}(\underline{G})$.

For the right-hand side, note first that cardinality preserves homotopy sums (i.e. transforms the integral into a sum over π_0 divided by symmetry factors). Now $(D_1)_w \simeq C_1^w$ (as an object over D_1) has cardinality $\underline{G}^w$, and G_w has cardinality $\underline{G}_w$. In conclusion, homotopy cardinality of the main groupoid equivalence yields the Faà di Bruno formula at the algebraic level (where we no longer write underlines):

5.6. **Theorem.** *In the incidence bialgebra of a locally finite operad* R *we have*

$$\Delta(G) = \sum_w G^w \otimes G_w / \left|\mathrm{Aut}(w)\right| .$$

6. Generalisations

So far we have treated the case of the terminal algebra for an operad. Several generalisations follow readily by closer inspection of the constructions and proofs, as we proceed to explain.

6.1. **Non-discrete colours.** In the polynomial monads corresponding to operads, the groupoid of colours I is discrete. We wish to give up that condition, because there are interesting examples with non-discrete I, namely operads in groupoids (cf. 7.5 and 7.6 below). The only place where the discreteness condition was used, was in the proof of Proposition 3.5, where it ensured that the strictly cartesian 2-natural transformation $\phi : F_! \mathsf{R} \Rightarrow \mathsf{S} F_!$ is also homotopy cartesian. But in fact we needed this only for naturality squares on arrows of the type $F_! \alpha$ (and S applied to those). If instead we require $F_! \alpha$ to be a fibration, we can draw the same conclusion and establish Proposition 3.5 again, without discreteness assumptions. $F_! \alpha$ is precisely the map of groupoids $B \to I$ in the diagram defining R, so we impose now the condition that this is a fibration. (Note that it is always a fibration when I is discrete.)

6.2. **Algebras.** The second generalisation is to note that to set up the Segal space $D_\bullet$, we did not rely on the algebra $\alpha : \mathsf{R}A \to A$ being the terminal algebra. In fact, all the arguments in the construction work exactly the same for any (strict) R-algebra. We should still require, though, that $F_! \alpha$ is a fibration of groupoids (or that I is discrete).

6.3. **General S.** So far we have assumed that S is the symmetric monoidal category monad, so that monads R over it correspond to operads (and operads in groupoids). In fact, all the results generalise readily to any general finitary polynomial monad S (although for simplicity, we assume that the maps in the representing polynomial diagram are fibrations): the general situation concerns any cartesian monad morphism between polynomial monads

$$\mathsf{R} \overset{F}{\Rightarrow} \mathsf{S}$$

and leads to R-algebras internal to categorical S-algebras (as developed by [**2, 3, 4, 48, 52**]). The fibrancy condition on (A, α) is still required, of course.

One thing that changes when S is no longer the symmetric monoidal category monad is that the category object $D_\bullet$ is no longer symmetric monoidal but is instead a categorical S-algebra, now for the general S. Each choice of S will give a new ambient setting, and a new notion of connectedness: the connected Green function will always be given by the unit of the monad.

Since the proofs in Section 2 did not actually use other properties of the monad S than being polynomial, cartesian and homotopy cartesian, we get immediately in this more general situation:

Proposition 6.4. *$\boldsymbol{Grpd}_{/D_1}$ is an S-algebra in $\boldsymbol{Coalg}(\boldsymbol{LIN})$.*

Accordingly, taking cardinality will yield an S-algebra object in the category of coalgebras, or, equivalently, a comonoid in the category of S-algebras.

In the operad case, the setting for the Faà di Bruno formula is a free algebra (or more precisely a power-series ring), and it is straightforward to interpret the exponent w in the left-hand factor G^w: it is simply G multiplied with itself w times, where in reality w represents the shape of a list of things to be multiplied. For general S, the exponent w represents the shape of an S-configuration, and G^w denotes such a configuration of elements in G, which can be multiplied using the (free) S-algebra structure. The precise meaning of the exponent is given in 3.12, and hopefully the examples will clarify this point.

The most obvious alternative to the symmetric monoidal category monad is to take S to be the (nonsymmetric) monoidal category monad, which amounts to considering nonsymmetric operads. The outcome are noncommutative bialgebras, and some kind of noncommutative Faà di Bruno formula. See Example 7.7 below.

7. Examples

In the first few examples we maintain as S the symmetric monoidal category monad. Hence the bialgebras will be free commutative. We also keep a discrete groupoid of colours.

7.1. **Classical Faà di Bruno.** Take R to be S_+, the polynomial monad for commutative semi-monoids. It is like S, but omitting the nullary operation. It is the terminal reduced operad Comm_+ (the word 'positive' is also used instead of 'reduced' [**1**]). One can check that the bar category $D_\bullet$ is then the fat nerve (see [**23**, 2.14]) of the category of finite sets and surjections. Indeed, D_0 is $\mathsf{S}1$, which is equivalent to the groupoid of finite sets and bijections—this is also degree 0 of the fat nerve. Next, D_1 is $\mathsf{S}\mathsf{S}_+1$, the groupoid of symmetric lists of nonempty symmetric lists of 1s, naturally equivalent with the groupoid of surjections, degree 1 of the fat nerve. The same identifications work for general n. This fat nerve is symmetric monoidal under disjoint union, and the resulting bialgebra is the classical Faà di Bruno bialgebra (this observation goes back to [**31**]; see [**25**] for an elaboration), and the resulting Faà di Bruno formula is the classical one.

7.2. **Multivariate Faà di Bruno.** We consider the multi-variate version of the previous example. Let I be a set of colours. Let R be the operad whose n-ary operations are $(n+1)$-tuples of colours (and without nullary operations). The symmetries are colour-preserving bijections respecting the base point. One may think of these operations as corollas whose leaves and root are decorated with elements in I. The substitution operation takes a two-level trees with decorated edges and contracts the inner edges, simply forgetting their colours. Note that since the inner-edge colours are simply lost in this process, many substitutions give the same result. For this reason, in order for R to be locally finite we must demand I to be a finite set.

The bar construction of R is the category object in groupoids with D_0 the free groupoid on I, that is, tuples of colours and colour-preserving bijections. D_1 is the groupoid whose elements are coloured surjections, and whose arrows are pairs of colour-preserving bijections. We point out that this Segal groupoid is not Rezk complete (see [**24**] for discussion of this issue): for any two colours i and j there is a unary operation $i \to j$, with inverse $j \to i$. These invertible unaries do not come from D_0, which only contains colour preserving bijections.

The bialgebra resulting from $D_\bullet$ is the polynomial algebra generated by symbols $A_{w,i}$ with $i \in I$ and w a nonempty word in I, one generator for each iso-class of connected coloured surjections. Closely related to the non-Rezk-ness of $D_\bullet$ is the fact that the identity surjections are not group-like. Indeed, we have

$$\Delta(A_{i,i}) = \sum_{j \in I} A_{i,j} \otimes A_{j,i}$$

expressing that the identity $i \to i$ admits factorisations $i \to j \to i$. This bialgebra is precisely the dual to composition of multi-variate power series; more precisely I-tuples of power series in I-many variables. (The linear (unary) part of this substitution is simply matrix multiplication, and we recognise indeed the above formula as dual to the formula for the ith diagonal entry in a matrix product.)

7.3. **Algebras.** Continuing the case $\mathsf{R} = \mathsf{S}_+$, let A be an S_+-algebra, i.e. a commutative semi-monoidal groupoid. As in the two previous examples, the resulting bar construction $D_\bullet$ is a certain groupoid-enriched category of decorated surjections, but this time the decorations on each corolla are by elements in A such that the output colour is the semimonoid product of all the input colours. In particular, there are no unary operations $i \to j$ for $i \neq j$, and it follows readily that $D_\bullet$ is Rezk complete this time, just as in 7.1.

The coalgebra structure does not change much from that in 7.1, because only the underlying surjection really matters: the possible decorations in the factorisations are completely determined by the decorations of the original surjection.

In the next three examples, we give up the requirement that I is discrete.

7.4. **Free operads (Faà di Bruno for trees [21]).** Let P be any finitary polynomial endofunctor, and let R denote the free monad on P. Its operations are the P-trees. Assuming that the groupoid $P1$ is locally finite, also the groupoid of P-trees is locally finite, and the factorisations of operations amount to cuts in trees (as in [**35**]), and since a given tree admits only finitely many cuts, it follows that R is locally finite. The resulting bialgebra is the P-tree version ([**35**]) of the Butcher–Connes–Kreimer Hopf algebra [**12**] (but note that it is a bialgebra not a Hopf algebra: it fails to be connected for the node grading, because all the nodeless forests are of degree 0). The corresponding Faà di Bruno formula is that from [**21**].

The *core* of a P-tree is the combinatorial tree obtained by forgetting the P-decoration and shaving off leaves and root [**35**]. Taking core defines a bialgebra homomorphism from the bialgebra of P-trees to the usual Butcher–Connes–Kreimer Hopf algebra, thereby producing Faà di Bruno sub-Hopf algebras, including virtually all the ones of Bergbauer and Kreimer [**5**] (but not those of Foissy [**20**], whose grading is of a different nature); (see [**37**] for these results).

In the next example, it is interesting also to consider arbitrary algebras.

7.5. **Node substitution in trees: the monad for operads.** Symmetric operads are themselves algebras for a polynomial monad R; it is given by

$$\mathbb{B} \leftarrow \mathbb{T}^* \rightarrow \mathbb{T} \rightarrow \mathbb{B}$$

where (as usual) $\mathbb{B}$ is the groupoid of finite sets and bijections, $\mathbb{T}$ is groupoid of rooted (operadic) trees, and $\mathbb{T}^*$ is the groupoid of rooted (operadic) trees with a marked node. The map $\mathbb{B} \leftarrow \mathbb{T}^*$ returns the set of incoming edges of the marked node, $\mathbb{T}^* \rightarrow \mathbb{T}$ forgets the mark, and $\mathbb{T} \rightarrow \mathbb{B}$ returns the set of leaves. An R-algebra is thus a map $A \rightarrow \mathbb{B}$ (which should be required to be a discrete fibration in order to get the usual notion of $\boldsymbol{Set}$-operad), whose (homotopy) fibre over $n \in \mathbb{B}$ is thought of as the set of n-ary operations. $\mathsf{R}A$ is the groupoid of trees whose nodes are decorated by operations in A, and the monad structure $\alpha : \mathsf{R}A \rightarrow A$ gives precisely the operad substitution law, prescribing how to contract a whole tree configuration of operations to a single operation.

To ensure local finiteness of R, one should exclude the trivial tree, and the resulting notion of operad is then that of non-unital operad.

The incidence bialgebra of R, corresponding to the terminal operad, is the slice $\boldsymbol{Grpd}_{/\mathbb{T}}$ but with a comultiplication different from that in Example 7.4: a tree t is comultiplied

$$\Delta(t) = \sum_{\text{subtree covers}} \textstyle\prod_i s_i \otimes q$$

by summing over all ways to cover the tree with subtrees s_i, disjoint on nodes, then interpreting those subtrees as a forest $\prod_i s_i$ (the left-hand tensor factor), and contracting each subtree s_i to a corolla to obtain a quotient tree q (the right-hand tensor factor).

Taking cores constitutes a bialgebra homomorphism to the Hopf algebra of trees of Calaque–Ebrahimi-Fard–Manchon [**7**], which is of interest since it governs substitution of Butcher series in numerical analysis [**10**]. Our general result now gives a Faà di Bruno formula in the incidence bialgebra, and via the core homomorphism a Faà di Bruno sub-bialgebra in the Calaque–Ebrahimi-Fard–Manchon Hopf algebra, which seems not to have been noticed before.

For general R-algebras A, i.e. operads, the constructions and descriptions are the same, except that the trees are now A-trees, i.e. trees decorated with A-operations on the nodes. The bialgebra has the same description again, but it should be noted that contracting a subtree to a corolla, and still obtain an operation to decorate the resulting node with, involves the operad structure of A.

7.6. **Feynman graphs.** The 1PI connected Feynman graphs for a given quantum field theory form the operations of an operad in groupoids [**38**], which we now take as R: let I denote the groupoid of interaction labels (connected graphs without internal lines), let $\mathbb{G}$ denote the groupoid of all 1PI connected Feynman graphs with residue in I, and let $\mathbb{G}^*$ denote the groupoid of all such graphs, but with a marked vertex. The polynomial representing the operad R is

$$I \xleftarrow{s} \mathbb{G}^* \xrightarrow{p} \mathbb{G} \xrightarrow{t} I$$

where s returns the marked vertex, p forgets the marking, and t returns the residue of the graph. The monad multiplication is given by substitution of graphs into vertices. The resulting bialgebra is the bialgebra version [**36**] of the Connes–Kreimer Hopf algebra of graphs [**13**]. The connected Green function is the standard (bare) combinatorial Green function in quantum field theory [**5**], and the Faà di Bruno formula is a non-renormalised version of the formula of van Suijlekom [**47**].

We now pass to the noncommutative setting: S is now the monoidal category monad, so that polynomial monads over it are nonsymmetric operads. The resulting bialgebras at the vector-space level are now free noncommutative.

7.7. **The noncommutative Faà di Bruno bialgebra.** Let R be the reduced part of S, i.e. the semimonoid monad. The polynomials representing these monads are

$$\begin{tikzcd}
1 \arrow[d] & \mathbb{N}'_{>0} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{N}_{>0} \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 & \mathbb{N}' \arrow[l] \arrow[r] & \mathbb{N} \arrow[r] & 1
\end{tikzcd}$$

where the set of natural numbers $\mathbb{N}$ is a skeleton of the groupoid of finite ordered sets and monotone bijections, and $\mathbb{N}'$ is a skeleton of the groupoid of finite pointed ordered sets and basepoint-preserving monotone bijections. Hence the fibre of $\mathbb{N}' \to \mathbb{N}$ over n is the linear order $\underline{n}$.

The bar construction is the fat nerve of the category of linearly ordered finite sets and monotone surjections. Since linearly ordered sets have no automorphisms, this is equivalent to the strict nerve of any skeleton of the category. The bialgebra in this case is the (dual) Landweber–Novikov bialgebra from algebraic topology (see for example [**41**]), which is called the noncommutative Faà di Bruno bialgebra by Brouder, Frabetti and Krattenthaler [**6**]. It is also isomorphic (over $\mathbb{Q}$) to the Dynkin–Faà di Bruno bialgebra, introduced in the theory of numerical integration on manifolds by Munthe-Kaas [**42**]; see also [**16**]. In the connected Green function and in the Faà di Bruno formula, the homotopy sum $\int^k$ is now an ordinary sum $\sum_k$ since there are no symmetries present. The formula

$$\Delta(A) = \sum_k A^k \otimes A_k$$

is precisely the noncommutative Faà di Bruno formula of [**6**], modulo the shift in indexing mentioned in 1.6.

One may consider also a multivariate version of this noncommutative Faà di Bruno bialgebra, proceeding as in 7.2, but without symmetries. The case of two variables is treated in [**6**].

7.8. **Braided operads.** Take S to be the braided monoidal category monad, which is polynomial, represented by the classifying space of the braid groups. Taking R to be the commutative monoid monad (denoted S in the earlier sections), we obtain a braided version of the Faà di Bruno bialgebra, corresponding to the simplicial groupoid $D_\bullet$ whose codescent object is the category of vines [**52**]. The braiding is only manifest at the groupoid level though, as taking cardinality turns the braided monoidal structure into a commutative monoid.

To finish we consider a few silly examples of S just to illustrate different instances of the Faà di Bruno formula in certain degenerate situations.

7.9. **Two rather trivial examples.** Let S be the identity monad. A monad cartesian over S is then precisely a small category. The finiteness condition is then for that category to be locally finite.

The S-algebra structure is void, so the result of the construction is the usual incidence coalgebra of the category [**23**]. The connected Green function G is then simply the sum of all arrows (everything is connected). We have $G = \sum_k g_k$ where k runs over all the objects and g_k denotes the set of arrows with domain k. The Faà di Bruno formula reads

$$\Delta(G) = \sum_k G^k \otimes g_k.$$

Here G^k is the groupoid of 1-tuples of arrows with codomain k.

In the special case where R is just a monoid (i.e. a category with only one object), then the finiteness condition amounts to the finite-decomposition property of Cartier–Foata [**9**], and the Faà di Bruno formula reduces to

$$\Delta(G) = G \otimes G$$

—the connected Green function is group-like in this case.

As a slight elaboration on this example, take S to be the pointed set monad. This will yield a pointed comonoid. The resulting Faà di Bruno formula has only two terms.

Let's take the one-object case. A monad over S is a monoid M together with a left module E. One can think of this as an operad with only nullary operations E and unary operations M. The generators for the pointed coalgebra is the set $D_1 = 1 + E + M$. We have $\Delta(1) = 1 \otimes 1$, and for each $e \in E$ we have $\Delta(e) = 1 \otimes e + \sum_{xm=e} x \otimes m$ (with $x \in E$ and $m \in M$). Finally for $a \in M$ we have $\Delta(a) = \sum_{nm=a} n \otimes m$.

Inside D_1, the connected Green function is given by $G = E + M \subset 1 + E + M$. We have

$$\Delta(G) = 1 \otimes E + G \otimes M.$$

In a tree interpretation, this says that the 'cuts' are either empty-followed-by-nullary or anything-followed-by-unary.

Acknowledgements. We are indebted to Imma Gálvez and Andy Tonks for many discussions on the Faà di Bruno formula over the years—we owe of course a lot to [**21**]–[**25**]—and more specifically for help with Example 7.7. J.K. was supported by grant number MTM2013-42293-P of Spain. M.W. acknowledges the support of the Australian Research Council grant No. DP130101172, and the Institute of Mathematics of the Czech Academy of Sciences.

References

[1] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, vol. 29 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown, Stephen Chase and André Joyal.

[2] John C. Baez and James Dolan. *Higher-dimensional algebra. III. n-categories and the algebra of opetopes.* Adv. Math. **135** (1998), 145–206. ArXiv:q-alg/9702014.

[3] Michael Batanin. *The Eckmann-Hilton argument and higher operads.* Adv. Math. **217** (2008), 334–385. ArXiv:math/0207281.

[4] Michael Batanin and Clemens Berger. *Homotopy theory for algebras over polynomial monads.* Preprint, arXiv:1305.0086.

[5] Christoph Bergbauer and Dirk Kreimer. *Hopf algebras in renormalization theory: locality and Dyson-Schwinger equations from Hochschild cohomology.* In *Physics and number theory*, vol. 10 of IRMA Lect. Math. Theor. Phys., pp. 133–164. Eur. Math. Soc., Zürich, 2006. ArXiv:hep-th/0506190.

[6] Christian Brouder, Alessandra Frabetti, and Christian Krattenthaler. *Non-commutative Hopf algebra of formal diffeomorphisms.* Adv. Math. **200** (2006), 479–524. ArXiv:math/0406117.

[7] Damien Calaque, Kurusch Ebrahimi-Fard, and Dominique Manchon. *Two interacting Hopf algebras of trees: a Hopf-algebraic approach to composition and substitution of B-series.* Adv. Appl. Math. **47** (2011), 282–308. ArXiv:0806.2238.

[8] Louis Carlier and Joachim Kock. *Homotopy theory and combinatorics of groupoids.* In preparation (2016).

[9] Pierre Cartier and Dominique Foata. *Problèmes combinatoires de commutation et réarrangements.* No. 85 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, New York, 1969. Republished in the "books" section of the Séminaire Lotharingien de Combinatoire.

[10] Philippe Chartier, Ernst Hairer, and Gilles Vilmart. *Algebraic structures of B-series.* Found. Comput. Math. **10** (2010), 407–427.

[11] J. R. B. Cockett and R. A. G. Seely. *The Faà di Bruno construction.* Theory Appl. Categ. **25** (2011), 394–425.

[12] Alain Connes and Dirk Kreimer. *Hopf algebras, renormalization and noncommutative geometry.* Comm. Math. Phys. **199** (1998), 203–242. ArXiv:hep-th/9808042.

[13] Alain Connes and Dirk Kreimer. *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem.* Comm. Math. Phys. **210** (2000), 249–273. ArXiv:hep-th/9912092.

[14] Peter Doubilet. *A Hopf algebra arising from the lattice of partitions of a set.* J. Algebra **28** (1974), 127–132.

[15] Luis A. Duffaut Espinosa, Kurusch Ebrahimi-Fard, and W. Steven Gray. *A combinatorial Hopf algebra for nonlinear output feedback control systems.* J. Algebra **453** (2016), 609–643. ArXiv:1406.5396.

[16] Kurusch Ebrahimi-Fard, Alexander Lundervold, and Dominique Manchon. *Noncommutative Bell polynomials, quasideterminants and incidence Hopf algebras.* Internat. J. Algebra Comput. **24** (2014), 671–705. ArXiv:1402.4761.

[17] Kurusch Ebrahimi-Fard and Frédéric Patras. *Exponential renormalization.* Ann. Henri Poincaré **11** (2010), 943–971. ArXiv:1003.1679.

[18] Héctor Figueroa and José M. Gracia-Bondía. *Combinatorial Hopf algebras in quantum field theory. I.* Rev. Math. Phys. **17** (2005), 881–976. ArXiv:hep-th/0408145.

[19] Thomas M. Fiore, Nicola Gambino, and Joachim Kock. *Monads in double categories.* J. Pure Appl. Algebra **215** (2011), 1174–1197. ArXiv:1006.0797.

[20] Loïc Foissy. *Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson-Schwinger equations.* Adv. Math. **218** (2008), 136–162. ArXiv:0707.1204.

[21] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Groupoids and Faà di Bruno formulae for Green functions in bialgebras of trees.* Adv. Math. **254** (2014), 79–117. ArXiv:1207.6404.

[22] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Homotopy linear algebra.* To appear in Proc. Royal Soc. Edinburgh A (2016). ArXiv:1602.05082.

[23] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces, incidence algebras and Möbius inversion I: basic theory.* Preprint, arXiv:1512.07573.

[24] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces, incidence algebras and Möbius Inversion II: completeness, length filtration, and finiteness.* Preprint, arXiv:1512.07577.

[25] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces in combinatorics*. In preparation. For the moment, the results quoted can be found in Section 5 of arXiv:1404.3202.

[26] Nicola Gambino and Joachim Kock. *Polynomial functors and polynomial monads*. Math. Proc. Cambridge Phil. Soc. **154** (2013), 153–192.

[27] W. Steven Gray and Luis A. Duffaut Espinosa. *A Faà di Bruno Hopf algebra for a group of Fliess operators with applications to feedback*. Systems & Control Letters **60** (2011), 441–449.

[28] Fred M. Hoppe. *Faà di Bruno's formula and the distributions of random partitions in population genetics and physics*. Theoretical Population Biology **73** (2008), 543–551.

[29] Warren P. Johnson. *The curious history of Faà di Bruno's formula*. Amer. Math. Monthly **109** (2002), 217–234.

[30] Saj-nicole A. Joni and Gian-Carlo Rota. *Coalgebras and bialgebras in combinatorics*. Stud. Appl. Math. **61** (1979), 93–139.

[31] André Joyal. *Une théorie combinatoire des séries formelles*. Adv. Math. **42** (1981), 1–82.

[32] Maurice G. Kendall. *The Advanced Theory of Statistics*. Vol. I. J. B. Lippincott Co., Philadelphia, 1944.

[33] Joachim Kock. *Polynomial functors and trees*. Int. Math. Res. Notices **2011** (2011), 609–673. ArXiv:0807.2874.

[34] Joachim Kock. *Data types with symmetries and polynomial functors over groupoids*. In *Proceedings of the 28th Conference on the Mathematical Foundations of Programming Semantics (Bath, 2012)*, vol. 286 of Electronic Notes in Theoretical Computer Science, pp. 351–365, 2012. ArXiv:1210.0828.

[35] Joachim Kock. *Categorification of Hopf algebras of rooted trees*. Cent. Eur. J. Math. **11** (2013), 401–422. ArXiv:1109.5785.

[36] Joachim Kock. *Perturbative renormalisation for not-quite-connected bialgebras*. Lett. Math. Phys. **105** (2015), 1413–1425. ArXiv:1411.3098.

[37] Joachim Kock. *Polynomial functors and combinatorial Dyson–Schwinger equations*. Preprint, arXiv:1512.03027.

[38] Joachim Kock. *Categorical formalisms for graphs and trees in quantum field theory*. In preparation; the quoted results have been previewed in [**34**] and [**37**]. (2016).

[39] Pierre Leroux. *Les catégories de Möbius*. Cahiers Topol. Géom. Différ. **16** (1976), 280–282.

[40] J. Peter May. *The geometry of iterated loop spaces*, vol. 271 of Lectures Notes in Mathematics. Springer-Verlag, Berlin, 1972.

[41] Jack Morava. *Some examples of Hopf algebras and Tannakian categories*. In *Algebraic topology (Oaxtepec, 1991)*, vol. 146 of Contemp. Math., pp. 349–359. Amer. Math. Soc., Providence, RI, 1993.

[42] Hans Munthe-Kaas. *Lie-Butcher theory for Runge-Kutta methods*. BIT **35** (1995), 572–587.

[43] Thomas J. Robinson. *New perspectives on exponentiated derivations, the formal Taylor theorem, and Faà di Bruno's formula*. In *Vertex operator algebras and related areas*, vol. 497 of Contemp. Math., pp. 185–198. Amer. Math. Soc., Providence, RI, 2009. ArXiv:0903.3391.

[44] Ross Street. *Fibrations and Yoneda's lemma in a 2-category*. Lecture Notes in Math. **420** (1974), 104–133.

[45] Ross Street. *The formal theory of monads*. J. Pure Appl. Algebra **2** (1972), 149–168.

[46] Walter D. van Suijlekom. *Renormalization of gauge fields: a Hopf algebra approach*. Comm. Math. Phys. **276** (2007), 773–798. ArXiv:hep-th/0610137.

[47] Walter D. van Suijlekom. *The structure of renormalization Hopf algebras for gauge theories. I. Representing Feynman graphs on BV-algebras*. Comm. Math. Phys. **290** (2009), 291–319. ArXiv:0807.0999.

[48] Mark Weber. *Operads within monoidal pseudo algebras*. Appl. Categ. Struct. **13** (2005), 389–420.

[49] Mark Weber. *Familial 2-functors and parametric right adjoints*. Theory Appl. Categ. **18** (2007), 665–732.

[50] Mark Weber. *Polynomials in categories with pullbacks*. Theory Appl. Categ. **30** (2015), 533–598. ArXiv:1106.1983.

[51] Mark Weber. *Operads as polynomial 2-monads*. Theory Appl. Categ. **30** (2015), 1659–1712. ArXiv:1412.7599.

[52] Mark Weber. *Internal algebra classifiers as codescent objects of crossed internal categories*. Theory Appl. Categ. **30** (2015), 1713–1792. ArXiv:1503.07585.

[53] Mark Weber. *Algebraic Kan extensions along morphisms of internal algebra classifiers*. Tbilisi Math. J. **9** (2016), 65–142.

Departament de matemàtiques, Universitat Autònoma de Barcelona
E-mail address: kock@mat.uab.cat

Institute of Mathematics of the Czech Academy of Sciences
E-mail address: mark.weber.math@gmail.com