Operads within monoidal pseudo algebras

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ABSTRACT. A general notion of operad is given, which includes:

(1) the operads that arose in algebraic topology in the 1970's to characterise loop spaces.
(2) the higher operads of Michael Batanin [**Bat98**].
(3) braided and symmetric analogues of Batanin's operads which are likely to be important in the study of weakly symmetric higher dimensional monoidal categories.

The framework of this paper, links together 2-dimensional monad theory, operads, and higher dimensional algebra, in a natural way.

1. Introduction

Operads arose first in the early 1970's in algebraic topology [**BV73**] [**May72**] to keep track of the combinatorial data that characterises infinite loop spaces.[1] In the most basic situation, one has a braided monoidal category $(\mathcal{V}, I, \otimes)$, and defines an operad to be a sequence of objects $(p_n : n \in \mathbb{N})$ of $\mathcal{V}$, together with maps

$$I \to p_1$$

$$p_k \otimes (p_{n_1} \otimes ... \otimes p_{n_k}) \to p_n$$

where $n = \sum_i n_i$. This data satisfies some axioms, that ensure that it is sensible to regard each object p_n, as an object of n-ary operations, and the maps as expressing the process of substitution of operations. The p_n and the corresponding maps are called a *non-symmetric operad* within the braided monoidal category $(\mathcal{V}, I, \otimes)$. For applications, $\mathcal{V}$ can be some category of spaces, chain complexes, differential graded algebras, or simplicial sets. Typically, $\mathcal{V}$ is actually a symmetric monoidal category, one has symmetric group actions on each p_n, and asks that these actions be compatible with the substitution. Such an operad is known as a symmetric operad within a symmetric monoidal category.

Beginning with insights of Todd Trimble [**Tri**], and then in the work of Michael Batanin [**Bat98**], operads were shown to be fundamental for the explicit combinatorial description of higher dimensional categorical structures. However, the operads used in higher dimensional algebra are typically somewhat more intricate than those originally conceived to characterise loop spaces, although the basic idea of formalising some notion of substitution remains the same.

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[1] For more on the history of operads and their recent resurgence in modern mathematics see [**MMS02**].

In [**Bat98**] as part of an approach to defining weak ω-categories, Batanin conceived of a notion of *higher operad* internal to a structure he called an augmented monoidal globular category. This new operad notion is more complicated for two reasons. First, monoidal categories are replaced by the more complicated augmented monoidal globular categories. Second, natural numbers n as the place holders of the objects p_n of the sequence, are replaced by trees. Just as addition of natural numbers may be regarded as a consequence of the notion of monoidal category in that $\mathbb{N}$ with its addition is the strict monoidal category freely generated by one object, trees and their arithmetic operations (pasting of trees), are encapsulated by the notion of monoidal globular category.

The notion of operad defined in this paper formalises this phenomenon in the following way. One begins with a 2-monad T on a 2-category $\mathcal{K}$ whose job is twofold:

(1) To describe the external structure within which the corresponding operads live. For example, to define non-symmetric operads one takes T to be the 2-monad $\mathcal{M}$ on **CAT** whose algebras are monoidal categories. Non-symmetric operads live inside braided monoidal categories, which are expressed here as *monoidal pseudo algebras* for the 2-monad $\mathcal{M}$.
(2) To encapsulate the "indexing type". For example a sequence p of objects in $\mathcal{V}$ is nothing but a functor $p : \mathcal{M}(1) \to \mathcal{V}$ because $\mathcal{M}(1) = \mathbb{N}$. The monad structure of $\mathcal{M}$ expresses the addition of natural numbers, which is necessary for the definition of non-symmetric operad.

An operad is then defined relative to T. In this way a unified formalism for the operads originally considered in algebraic topology and those of interest to higher dimensional algebra is achieved, with different operad notions obtained by varying T.

The idea central to this definition, is to regard the external structure as composed of two parts: a pseudo algebra structure for the 2-monad, together with a compatible pseudo monoid structure. Taken together one has the notion of monoidal pseudo algebra described in this paper. The origin of this idea is in the observation that when one describes the substitution maps for a non-symmetric operad

$$p_k \otimes (p_{n_1} \otimes ... \otimes p_{n_k}) \to p_n,$$

there are really two different types of tensor product at work. One has a binary tensor product as in $p_k \otimes (...)$, and k-ary tensor products as in $p_{n_1} \otimes ... \otimes p_{n_k}$. The binary tensor product is formalised as the pseudo monoid structure, and the k-ary tensor products are formalised as the pseudo $\mathcal{M}$-algebra structure. Their compatibility implies that they can be identified (as is the usual custom), and that the resulting monoidal structure is braided. The braiding is necessary for the expression of one of the operad axioms (associativity of substitution).

The study of higher-dimensional braids and tangles, as well as the homotopy groups of spheres, motivates the consideration of the various notions of monoidal n-category. A k-tuply monoidal weak n-category is a weak $(n+k)$-category with one cell in each dimension less than k. Such a structure is considered as being n-dimensional by reindexing appropriately, that is, by regarding the m-cells (for $m \geq k$) of the original weak $(n+k)$-category as $(m-k)$-cells in this new structure. For example, a 2-tuply monoidal tricategory is a braided monoidal category, and a braiding is a subtler notion of symmetry for monoidal categories than the usual one. Thus, one expects these k-tuply monoidal weak n-categories in general, to be higher dimensional monoidal categories which possess still more subtle symmetry.

Motivated by insights from homotopy theory, [**BD98**] give three hypotheses relating such structures to quantum topology. The first is that the n-cells of the free k-tuply monoidal weak n-category on one object correspond to "n-braids in $(n+k)$ dimensions", which are certain n-dimensional surfaces embedded in the $(n+k)$-dimensional cube. The case $n=1$ and $k=2$ gives the usual definition of braid, which corresponds to the morphisms of the free braided monoidal category on one object. Second is the corresponding hypothesis for weak n-groupoids, which relates to the fundamental n-groupoid of the k-fold loop space of the k-sphere. Finally, it is predicted that the n-cells of the free k-tuply monoidal weak n-category *with duals* correspond to "framed n-tangles in $(n+k)$ dimensions", which again are certain n-dimensional surfaces embedded in the $(n+k)$-dimensional cube. This time the case $n=1$ and $k=2$ corresponds to the usual definition of tangle which has been shown to correspond to the free braided monoidal category with duals on one object.

An important motivation for this work is to define braided and symmetric analogues of higher operads, to facilitate the study of these weakly symmetric higher dimensional categories. With the general operad definition at our disposal, this problem is reduced to finding appropriate 2-monads, which blend together the combinatorics of higher operads with braids and symmetries in a natural way. The 2-monad that parametrises Batanin's higher operads is denoted by $\mathcal{T}$ and acts on the 2-category $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}]$ of globular categories. Moreover there are 2-monads $\mathcal{B}$ and $\mathcal{S}$ on $\mathbf{CAT}$ which parametrise braided and symmetric operads in the usual sense.

The appropriate 2-monads alluded to above are obtained by regarding $\mathcal{B}$ and $\mathcal{S}$ as 2-monads on $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}]$ in an obvious way, and seeing that there are distributive laws between these 2-monads and $\mathcal{T}$. The existence of these distributive laws is deduced from an alternative description of the category ω-$\mathbf{Cat}$, of strict ω-categories, due to Clemens Berger [**Ber02**].

As observed in [**Lei03**] and [**Str00**], the higher operads which are actually used in [**Bat98**] to define weak ω-categories, all live in a particular augmented monoidal globular category called **Span**, and admit a far simpler description. One has the monad on $\mathcal{T}_0$ on $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ the category of globular sets whose algebras are strict ω-categories, and a higher operad in **Span** amounts to a cartesian monad morphism $\phi_0 : R_0 \to \mathcal{T}_0$. That is, a monad R_0 on $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, and a natural transformation ϕ_0 which is compatible with the monad structures, and whose naturality squares are pullbacks. We call such higher operads *basic higher operads*. On the other hand, [**Bat02**] uses the full generality of the higher operad notion for applications to loop spaces. So, while basic higher operads suffice for the definition of weak ω-category presented in [**Bat98**], it seems that general higher operads are important for applications.

In this paper we must speak of two set theoretic universes $\mathcal{U}_1 \in \mathcal{U}_2$ and distinguish between **Set**, the category of $\mathcal{U}_1$ small sets and functions between them, and **SET**, the category of $\mathcal{U}_2$ small sets. Similarly we distinguish between the corresponding 2-categories **Cat** and **CAT** of categories. So **Set** and **Cat** may be regarded as objects of **CAT**, as may many of the other categories that one encounters in applications of operads: categories of spaces, chain complexes, differential graded algebras and simplicial sets. The reason for this distinction is that the 2-monads T on 2-categories $\mathcal{K}$ that parametrise operad notions, apply at the $\mathcal{U}_2$ level. For example, the monad $\mathcal{M}$ on **CAT** that parametrises non-symmetric operads within braided monoidal categories. The braided monoidal categories within which our operads live are objects of **CAT**.

Having made this distinction, it is worth noting on the other hand, that most general categorical and combinatorial constructions do not depend on such considerations, that is, they are "universe-insensitive". This is part of the reason why such size issues are often glossed over. However in [**Str00**] such distinctions are shown to be pertinent to the organisation of the combinatorics and category theory which underlies higher dimensional algebra.[2] As part of such distinctions, we have used the notation $\phi_0 : R_0 \to \mathcal{T}_0$ to denote a basic higher operad, which is a morphism of monads on $[\mathbb{G}^{\text{op}}, \mathbf{Set}]$. The universe insensitivity alluded to above means that ϕ_0 could be regarded instead as a monad morphism on $[\mathbb{G}^{\text{op}}, \mathbf{SET}]$, and then taking internal category objects (which is 2-functorial) induces what we denote by $\phi : R \to \mathcal{T}$ – the corresponding 2-monad morphism on $[\mathbb{G}^{\text{op}}, \mathbf{CAT}]$. This notation is convenient for us, because the distributivity of braids and symmetries with $\mathcal{T}$ mentioned above, is actually more general – one can replace $\mathcal{T}$ with R for any basic higher operad ϕ_0. For example, R_0 could be a monad on $[\mathbb{G}^{\text{op}}, \mathbf{Set}]$ whose algebras are *weak* ω-categories in the sense of [**Bat98**]. In this way our formalism provides for each basic higher operad ϕ_0, a corresponding general notion of higher operad (as R operads in our sense), as well as braided and symmetric analogues of these.

This paper is organised as follows. Sections (2) and (3) review 2-monads and their algebras, and pseudo monoids, assuming familiarity with the usual categorical notions of monad and monoid. Monoidal pseudo algebras are introduced in section (4), and in section (5), operads and their algebras are defined in full generality. The examples presented in sections (2)–(5), taken together, exhibit how the conventional operad notions are captured by our formalism. Then in section (6), after briefly recalling the relevant background on the globular approach to higher category theory, the higher operads of [**Bat98**] are described as instances of our general operads. We begin section (7) by recalling the characterisation from [**Ber02**] of the category algebras of a basic higher operad. This is then re-expressed in the language of sketches, which then allows the easy explanation of the formal distributivity of symmetries and braids with basic higher operads.

2. 2-monads and pseudo algebras

In this section we recall the basic and well known notions and terminology of 2-dimensional monad theory, as well as the examples important to the present work.

Recall that a 2-monad (T, η, μ) on a 2-category $\mathcal{K}$ consists of an endo-2-functor T of $\mathcal{K}$, together with 2-natural transformations $\eta : 1 \Longrightarrow T$ and $\mu : T^2 \Longrightarrow T$,

[2] In [**Web**] these issues are discussed further, and following [**SW78**], [**Str74**] and [**Str80**], the relevant size issues are axiomatized.

called the unit and multiplication, so that

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

commute. We shall allow the usual abuse of referring to the 2-monad T, omitting reference to the unit and multiplication. Most of the examples of 2-monads of interest to us shall now be described, and for many more examples and further detail on 2-dimensional monad theory, the reader should consult [**BKP89**].

EXAMPLES 2.1. (1) Every monad (T, η, μ) on a category $\mathcal{E}$ can be regarded as a 2-monad, by regarding $\mathcal{E}$ as a locally discrete 2-category (that is, one with only identity 2-cells).

(2) For each $\mathcal{K}$ one obtains the identity monad $1_{\mathcal{K}}$ on $\mathcal{K}$, by taking T, η, and μ to be identities.

(3) Let $\mathcal{E}$ be a category with pullbacks, and (T, η, μ) be a monad on $\mathcal{E}$ such that T preserves pullbacks. One can then take $\mathcal{K}$ to be the 2-category $\mathbf{Cat}(\mathcal{E})$ of categories internal to $\mathcal{E}$. This process of taking the 2-category of internal categories is the object map of a 2-functor

$$\mathbf{PB} \xrightarrow{\mathbf{Cat}} \mathbf{2CAT}$$

from the 2-category of categories with pullbacks, pullback preserving functors and natural transformations between them, to the 2-category of 2-categories, 2-functors and 2-natural transformations. Applying this 2-functor, one obtains a 2-monad $\mathbf{Cat}(T)$ on $\mathbf{Cat}(\mathcal{E})$.

(4) As an instance of (3), take $\mathcal{E}$ to be $\mathbf{SET}$ and T the monoid monad on $\mathbf{SET}$. We denote by $\mathcal{M}$ the 2-monad $\mathbf{Cat}(T)$ on $\mathbf{CAT}$. An object of $\mathcal{M}(X)$ is a sequence of objects from X, that is, a functor $x : n \to X$ where $n \in \mathbb{N}$ is being regarded as the discrete category whose object set is $n = \{0, ..., n-1\}$. A morphism $f : x \to y$ in $\mathcal{M}(X)$ is a 2-cell

$$n \underset{y}{\overset{x}{\Downarrow f}} X \ ,$$

and so is just a sequence of maps in X. The 1 and 2-cell mappings for $\mathcal{M}$ are obtained by composition in the evident fashion. The unit for the monad picks out the sequences of length one, and the multiplication is given by concatenation of sequences.

(5) [**JS93**] Denote by $\mathbf{Br}_n$ the n-th braid group. We shall denote by $\mathcal{B}$ the following 2-monad on $\mathbf{CAT}$. An object of $\mathcal{B}(X)$ is again a sequence of objects of X. A morphism between two sequences x and y of the same length n consists of a braid on n-strings, whose strings are labelled by arrows in X. More precisely, such a morphism consists of $\beta \in \mathbf{Br}_n$,

together with a 2-cell

$$\begin{array}{ccc} n & \xrightarrow{\overline{\beta}} & n \\ & \searrow^{x} \overset{f}{\Rightarrow} \swarrow^{y} & \\ & X & \end{array},$$

where $\overline{\beta}$ is the underlying permutation of β regarded as a functor between discrete categories. The 2-functoriality of $\mathcal{B}$ and the unit work as with $\mathcal{M}$. The multiplication is described by concatenation of sequences, and substitution of braids into braids in the evident way.

(6) [**Kel74**] [**JS93**] Denote by $\mathbf{Sym}_n$ the n-th symmetric group. We shall denote by $\mathcal{S}$ the following 2-monad on **CAT**. An object of $\mathcal{S}(X)$ is again a sequence of objects of X. A morphism between two sequences x and y of the same length n consists of a permutation on n-strings whose strings are labelled by arrows in X. More precisely, such a morphism consists of $\beta \in \mathbf{Sym}_n$, together with a 2-cell

$$\begin{array}{ccc} n & \xrightarrow{\beta} & n \\ & \searrow^{x} \overset{f}{\Rightarrow} \swarrow^{y} & \\ & X & \end{array},$$

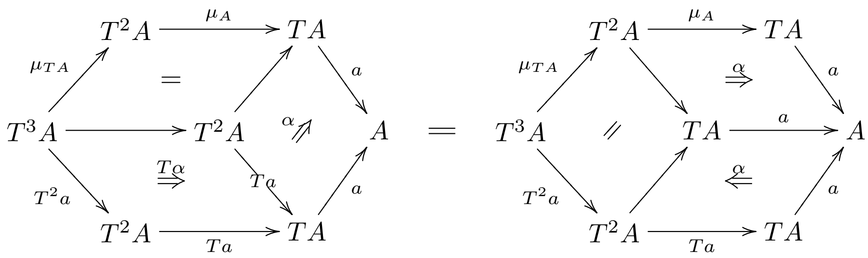
where β is being regarded as a functor between discrete categories. The 2-functoriality of $\mathcal{S}$ and the unit work as with $\mathcal{M}$. The multiplication is described by concatenation of sequences, and substitution of permutations into permutations in the evident way.

The important difference between 2-monads and ordinary monads on categories, is that there are various weaker notions of algebra in addition to the usual (Eilenberg-Moore) algebras for a monad. This makes 2-monad theory a natural choice of formalism when one wishes to consider coherently defined categorical structures. In this work we shall consider pseudo algebras and pseudo morphisms – where one replaces equality between composite arrows in the axiomatic definition of the objects and arrows of T-Alg, the category of Eilenberg-Moore algebras for T, by isomorphisms.

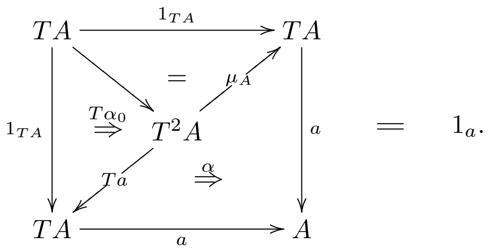
Definition 2.2. [**Str72**], [**Pow89**], [**CHP03**] Let (T, η, μ) be a 2-monad on a 2-category $\mathcal{K}$. A *pseudo T-algebra structure* (a, α_0, α) on an object $A \in \mathcal{K}$ consists of a 1-cell $a : TA \to A$ and invertible 2-cells

$$\begin{array}{ccc} T^2A & \xrightarrow{\mu_A} & TA \\ {\scriptstyle Ta}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow^{1_A} \overset{\alpha_0}{\Rightarrow} \swarrow^{a} & \\ & A & \end{array}$$

in $\mathcal{K}$ satisfying

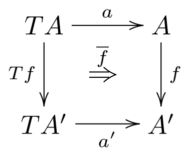


and

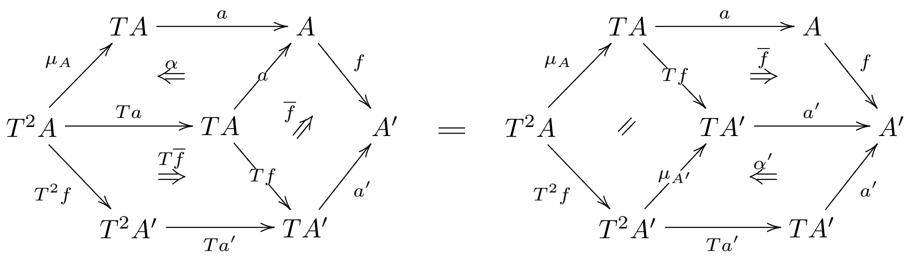


The triple (A, α_0, α) is referred to as a *pseudo T-algebra.* When α_0 is an identity the pseudo algebra is said to be *normal.* When in addition α is an identity, we refind the usual notion of T-algebra, and the algebra is said to be *strict.*

Definition 2.3. [**Str72**], [**Pow89**], [**CHP03**] Let (A, α_0, α) and (A', α'_0, α') be pseudo T-algebras. A *strong T-morphism structure* for a 1-cell $f : A \to A'$ is an invertible 2-cell



satisfying



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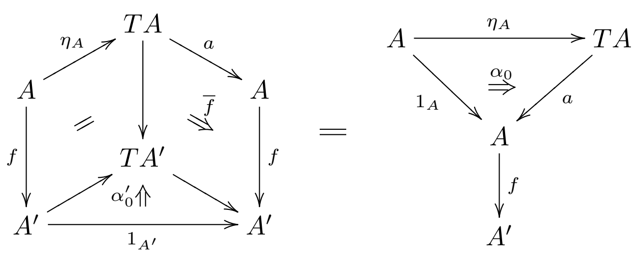
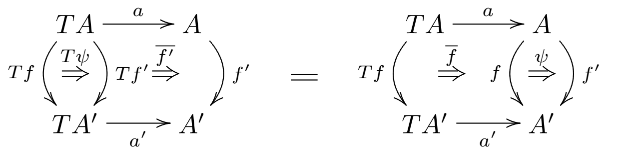


The pair $(f, \overline{f})$ is called a *strong T-morphism.* We shall allow the notational abuse of referring to the "strong T-morphism f", omitting any reference to $\overline{f}$. When $\overline{f}$ is an identity, we refind the usual notion of T-algebra morphism, and the T-morphism is said to be *strict* in this case.

Definition 2.4. [**Str72**], [**Pow89**], [**CHP03**] Let f and f' be strong T-morphisms $(a, \alpha_0, \alpha) \to (a', \alpha_0', \alpha')$. A 2-cell $\psi : f \Longrightarrow f'$ is an *algebra 2-cell* when



With the evident compositions, one defines the 2-category Ps-T-Alg to consist of pseudo T-algebras, strong T-morphisms and algebra 2-cells. The full sub-2-category of Ps-T-Alg consisting of the normal pseudo algebras is denoted Ps_0-T-Alg. The locally full sub-2-category of Ps-T-Alg consisting of the strict algebras and strict morphisms is denoted T-Alg_s.

Examples 2.5. (1) The 2-categories of strict and pseudo algebras coincide for (2.1)(1), being just the usual category of algebras for T regarded as a locally discrete 2-category.

(2) For any $\mathcal{K}$, a strict algebra structure for $1_{\mathcal{K}}$ is vacuous. A normal pseudo algebra structure is also vacuous. A pseudo algebra structure on $X \in \mathcal{K}$ amounts to $t : X \to X$ together with an isomorphism $t \cong 1_X$.

(3) The 2-category of strict algebras for (2.1)(3) is just $\mathbf{Cat}(T\text{-Alg})$.

(4) A strict $\mathcal{M}$-algebra structure on a category X is a strict monoidal structure. A pseudo $\mathcal{M}$-algebra structure on a category X is a monoidal structure described in an *unbiased* fashion. That is, one supplies an n-ary tensor product for $n \in \mathbb{N}$, and associated coherence isomorphisms. For a normal pseudo $\mathcal{M}$-algebra structure, the 1-ary tensor product of $x \in X$ is x, rather than just isomorphic to x. There are various monoidal coherence results in the literature, for example in [**Pow89**], [**Lac02**], [**Her00**] and [**Her01**], which are expressed in the language of pseudo algebras, and so apply to many other situations. In all these results, the inclusion 2-functor $\mathcal{M}$-$\mathrm{Alg}_s \to$Ps-$\mathcal{M}$-Alg is seen to have a left biadjoint for which the unit of the biadjunction is an equivalence. In addition to these results, one can also exhibit directly, a 2-equivalence between Ps_0-$\mathcal{M}$-Alg and the 2-category PsMon($\mathbf{CAT}$) consisting of monoidal categories defined in the usual (biased) way, by giving a binary tensor product and a unit object. Under this 2-equivalence, strong $\mathcal{M}$-morphisms coincide with the tensor functors of [**JS93**], and have been called strong monoidal functors elsewhere.

(5) A strict $\mathcal{B}$-algebra structure on a category X is a braided strict monoidal structure, that is, a braided tensor category in the sense of [**JS93**] whose underlying monoidal category is strict. A normal pseudo $\mathcal{B}$-algebra structure on a category X is a braided monoidal structure on X described in an *unbiased* fashion. As in the previous example there is a 2-equivalence between Ps_0-$\mathcal{B}$-Alg and the 2-category braided monoidal categories, braided monoidal functors and monoidal natural transformations defined as in [**JS93**].

(6) In the same way, strict $\mathcal{S}$ algebras are symmetric strict monoidal categories, normal pseudo $\mathcal{S}$-algebras are symmetric monoidal categories defined in an unbiased fashion, and these are equivalent to the usual notion of symmetric monoidal category.

3. Pseudo monoids

Having recalled a well-known "categorification" of monad algebra, we shall now recall one for the notion of monoid in a monoidal category. For us, it suffices to consider pseudo monoids within 2-categories with cartesian products in the **CAT**-enriched sense, rather than internal to a more general monoidal 2-category. Later, when we describe monoidal pseudo algebras and the operads they contain, this specialisation to 2-categories with *cartesian* products becomes crucial. For the remainder of this section, $\mathcal{K}$ is a 2-category with finite products. For the purposes of this section monoidal categories, strong monoidal functors and monoidal natural transformations correspond to the tensor categories, tensor functors and morphisms of tensor functors of [**JS93**].

Expressed 2-categorically, a monoidal category consists of data $(A, i, m, \alpha, \lambda, \rho)$ where A is a category, i and m are 1-cells

$$1 \xrightarrow{i} A \xleftarrow{m} A\times A$$

and (α, λ, ρ) are invertible natural transformations

$$\begin{array}{ccc} A\times A\times A & \xrightarrow{1\times m} & A\times A \\ {\scriptstyle m\times 1}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle m} \\ A\times A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{i\times 1} & A\times A & \xleftarrow{1\times i} & A \\ & {\scriptstyle 1}\searrow \; \overset{\lambda}{\Leftarrow} & \downarrow{\scriptstyle m} & \overset{\rho}{\Rightarrow} \; \swarrow{\scriptstyle 1} & \\ & & A & & \end{array}$$

satisfying the usual pentagon and unit axioms [**JS93**]. For any other category X, the category $\mathbf{CAT}(X, A)$ inherits a monoidal structure from A. Formally this is seen by applying $\mathbf{CAT}(X, -)$ to the data and axioms $(A, i, m, \alpha, \lambda, \rho)$, and noting that $\mathbf{CAT}(X, -)$ preserves products. Thus one obtains monoidal category structures on $\mathbf{CAT}(X, A)$ 2-naturally in X. In fact by the 2-categorical yoneda lemma this assignment of a natural family of monoidal category structures on $\mathbf{CAT}(X, A)$ from each monoidal structure on A, is bijective. The advantage of the representable definition is that it applies equally well in any 2-category with products.

Definition 3.1. [**JS93**] A *pseudo monoid structure* $(i, m, \alpha, \lambda, \rho)$ on $A \in \mathcal{K}$ consists of 1-cells

$$1 \xrightarrow{i} A \xleftarrow{m} A\times A$$

and invertible 2-cells

$$\begin{array}{ccc} A\times A\times A & \xrightarrow{1\times m} & A\times A \\ {\scriptstyle m\times 1}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle m} \\ A\times A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{i\times 1} & A\times A & \xleftarrow{1\times i} & A \\ & {\scriptstyle 1}\searrow \; \overset{\lambda}{\Leftarrow} & \downarrow{\scriptstyle m} & \overset{\rho}{\Rightarrow} \; \swarrow{\scriptstyle 1} & \\ & & A & & \end{array}$$

in $\mathcal{K}$ such that for all $X \in \mathcal{K}$, $(\mathcal{K}(X,i), \mathcal{K}(X,m), \mathcal{K}(X,\alpha), \mathcal{K}(X,\lambda), \mathcal{K}(X,\rho))$ is a monoidal structure for $\mathcal{K}(X,A)$. A *monoid* in $\mathcal{K}$ is a pseudo-monoid for which the two-cells in the above definition are identities.

Definition 3.2. [**JS93**] Let $(A, i, m, \alpha, \lambda, \rho)$ and $(A', i', m', \alpha', \lambda', \rho')$ be pseudo monoids. A *strong monoidal structure* for a 1-cell $f : A \to A'$ consists of invertible 2-cells

$$\begin{array}{ccccc} 1 & \xrightarrow{i} & A & \xleftarrow{m} & A\times A \\ {\scriptstyle 1}\downarrow & \overset{\phi_0}{\Leftarrow} & \downarrow{\scriptstyle f} & \overset{\phi_2}{\Rightarrow} & \downarrow{\scriptstyle f\times f} \\ 1 & \xrightarrow[i']{} & A' & \xleftarrow[m']{} & A'\times A' \end{array}$$

in $\mathcal{K}$ such that for all $X \in \mathcal{K}$, $(\mathcal{K}(X,f), \mathcal{K}(X,\phi_0), \mathcal{K}(X,\phi_2)$ provide the data for a strong monoidal functor $\mathcal{K}(X,A) \to \mathcal{K}(X,A')$. The strong monoidal morphism (f, ϕ_0, ϕ_2) is said to be strict, when ϕ_0 and ϕ_2 are identities.

Definition 3.3. [**JS93**] Let (f, ϕ_0, ϕ_2) and $(f', {\phi_0}', {\phi_2}')$ be strong monoidal morphisms. A 2-cell $\psi : f \Longrightarrow f'$ is a *monoidal* 2*-cell* when for all $X \in \mathcal{K}$, $\mathcal{K}(X,\psi)$ is a monoidal natural tranformation.

With the evident compositions, one defines the 2-category PsMon($\mathcal{K}$) to consist of pseudo monoids, strong monoidal morphisms and monoidal 2-cells. The locally full sub-2-category of PsMon($\mathcal{K}$) consisting of strict monoids and strict monoid morphisms is denoted as Mon($\mathcal{K}$). Note that the forgetful 2-functor

$$\mathrm{PsMon}(\mathcal{K}) \longrightarrow \mathcal{K}$$

can easily be seen to create products.

Example 3.4. The 2-category PsMon(**CAT**) consists of monoidal categories, strong monoidal functors, and monoidal natural transformations.

Examples 3.5. (1) Let $\mathcal{E}$ be a category with finite products, and let $\mathcal{K}$ be $\mathcal{E}$ regarded as a locally discrete 2-category. Then a pseudo monoid is just a monoid in $\mathcal{E}$ in the usual sense.

(2) Let $\mathcal{K}$ be PsMon(**CAT**). A pseudo monoid in $\mathcal{K}$ amounts to a braided monoidal category. More precisely, [**JS93**] provide a 2-equivalence between PsMon(PsMon(**CAT**)) and the usual 2-category of braided monoidal categories.

(3) Let $\mathcal{K}$ be PsMon(PsMon(**CAT**)). A pseudo monoid in $\mathcal{K}$ amounts to a symmetric monoidal category. More precisely, [**JS93**] provide a 2-equivalence between PsMon^3(**CAT**) and the usual 2-category of symmetric monoidal categories.

(4) Let $\mathcal{K}$ be PsMon^3(**CAT**). A pseudo monoid in $\mathcal{K}$ also amounts to a symmetric monoidal category. More precisely, [**JS93**] exhibit the forgetful 2-functor

$$\mathrm{PsMon}^4(\mathbf{CAT}) \longrightarrow \mathrm{PsMon}^3(\mathbf{CAT})$$

as a 2-equivalence.

By the representable definition of the 2-categories PsMon($\mathcal{K}$) and the 2-functoriality of PsMon, one immediately obtains the following "Eckmann Hilton" stabilisation result.

PROPOSITION 3.6. *Let $\mathcal{K}$ be a 2-category with finite products. Then the forgetful 2-functor*

$$\mathrm{PsMon}^{n+1}(\mathcal{K}) \longrightarrow \mathrm{PsMon}^{n}(\mathcal{K})$$

is a 2-equivalence when $n \geq 3$.

4. Monoidal pseudo algebras

For this section, let T be a 2-monad on a 2-category $\mathcal{K}$ with finite products. It is easily seen that both the forgetful 2-functors

$$\text{Ps-}T\text{-Alg}{\rightarrow}\mathcal{K} \qquad \text{Ps}_0\text{-}T\text{-Alg}{\rightarrow}\mathcal{K}$$

create products, and so in particular, Ps_0-T-Alg has finite products.

DEFINITION 4.1. A *monoidal pseudo T-algebra* is a pseudo monoid in Ps_0-T-Alg.

Unpacking this definition, one finds that a monoidal pseudo T-algebra consists of

- an object $A \in \mathcal{K}$.
- a normal pseudo T-algebra structure (a, α) on A.
- a pseudo monoid structure $(i, m, \beta, \lambda, \rho)$ on A.
- an invertible 2-cell $\overline{i}$ which provides i with a strong T-morphism structure.
- an invertible 2-cell $\overline{m}$ which provides m with a strong T-morphism structure.
- the 2-cells β, λ and ρ satisfy the T-algebra 2-cell axiom.

We shall refer to this monoidal pseudo algebra by the ordered 8-tuple $(A, a, i, m, \alpha, \beta, \lambda, \rho)$.

EXAMPLES 4.2. (1) For T as in (2.1)(1), a monoidal pseudo algebra is a monoid in T-Alg.
(2) For $T = 1_{\mathbf{CAT}}$ a monoidal pseudo algebra is a monoidal category. More generally, for $T = 1_{\mathcal{K}}$ as in (2.1)(2), a monoidal pseudo algebra is a pseudo monoid in $\mathcal{K}$.
(3) For $T = \mathcal{M}$ as in (2.1)(4), a monoidal pseudo algebra amounts to a braided monoidal category. More precisely we have a 2-equivalence

$$\mathrm{PsMon}(\mathrm{Ps}_0\text{-}\mathcal{M}\text{-Alg}) \simeq \mathrm{PsMon}(\mathrm{PsMon}(\mathbf{CAT}))$$

from (2.5)(4) and the 2-functoriality of PsMon, and by (3.5)(2) $\mathrm{PsMon}(\mathrm{PsMon}(\mathbf{CAT}))$ is 2-equivalent to the 2-category of braided monoidal categories. In this formalism the braiding arises from $\overline{m}$. In more detail, denote the object map of m by $m(x, y) = x {\otimes_0} y$, and the object map of a by $a(x_0, ..., x_{n-1}) = x_0 \otimes_1 ... \otimes_1 x_{n-1}$. Then $\overline{m}$ is an invertible 2-cell

$$\begin{array}{ccccc} \mathcal{M}(A{\times}A) & \longrightarrow & \mathcal{M}(A){\times}\mathcal{M}(A) & \xrightarrow{a \times a} & A{\times}A \\ {\scriptstyle M(m)}\downarrow & & \overset{\overline{m}}{\Rightarrow} & & \downarrow{\scriptstyle m} \\ \mathcal{M}(A) & & \xrightarrow[a]{\qquad\qquad\qquad} & & A \end{array}$$

where $\mathcal{M}(A\times A)\to\mathcal{M}(A)\times\mathcal{M}(A)$ is the canonical comparison. So the component of $\overline{m}$ at $((x_0,y_0),...,(x_{n-1},y_{n-1}))$ is an isomorphism

$$\begin{array}{c}(x_0\otimes_0 y_0)\otimes_1 ...\otimes_1(x_{n-1}\otimes_0 y_{n-1})\\ \downarrow \\ (x_0\otimes_1 ...\otimes_1 x_{n-1})\otimes_0(y_0\otimes_1 ...\otimes_1 y_{n-1})\end{array}$$

Writing I_0 for the unit for $\otimes_0$, the components of $\overline{i}$ are isomorphisms

$$\underbrace{I_0\otimes_1 ...\otimes_1 I_0}_{n}\to I_0$$

which in the case $n=0$, gives an isomorphism $I_1\cong I_0$, where I_1 is the unit for $\otimes_1$. Furthermore $x\otimes_1 y\cong x\otimes_0 y$ is obtained as:

$$\begin{array}{rclcl} x\otimes_1 y & \cong & (x\otimes_0 I_0)\otimes_1(I_0\otimes_0 y) & \cong & (x\otimes_1 I_0)\otimes_0(I_0\otimes_1 y) \\ & \cong & (x\otimes_1 I_1)\otimes_0(I_1\otimes_1 y) & \cong & x\otimes_0 y \end{array},$$

and a braiding in the usual sense is obtained as:

$$\begin{array}{rclcl} x\otimes_0 y & \cong & x\otimes_1 y & \cong & (I_0\otimes_0 x)\otimes_1(y\otimes_0 I_0) \\ & \cong & (I_0\otimes_1 y)\otimes_0(x\otimes_1 I_0) & \cong & (I_1\otimes_1 y)\otimes_0(x\otimes_1 I_1) \\ & \cong & y\otimes_0 x & & \end{array}.$$

These isomorphisms encode the Eckmann-Hilton argument (see [**Mac71**], pg 45, exercise 5).

(4) For $T=\mathcal{B}$ as in (2.1)(5), a monoidal pseudo algebra amounts to a symmetric monoidal category. More precisely we have a 2-equivalence $\mathrm{PsMon}(\mathrm{Ps}_0\text{-}\mathcal{B}\text{-Alg})\simeq\mathrm{PsMon}^3(\mathbf{CAT})$ from (2.5)(5), (3.5)(5) and the 2-functoriality of PsMon. By (3.5)(3), $\mathrm{PsMon}^3(\mathbf{CAT})$ is 2-equivalent to the 2-category of symmetric monoidal categories. The more explicit analysis here only differs from the previous example in that the action a already carries the information of a braiding for $\otimes_1$. The naturality of $\overline{m}$ ensures that the braiding encoded by it coincides with that described by a, and forces it to be a symmetry.

(5) For $T=\mathcal{S}$ as in (2.1)(5), a monoidal pseudo algebra amounts to a symmetric monoidal category. Arguing as in the previous examples we have a 2-equivalence $\mathrm{PsMon}(\mathrm{Ps}_0\text{-}\mathcal{B}\text{-Alg})\simeq\mathrm{PsMon}^4(\mathbf{CAT})$, and one between $\mathrm{PsMon}^4(\mathbf{CAT})$ and the usual 2-category of symmetric monoidal categories. The more explicit analysis here only differs from the previous example in that the action a already carries the information of a symmetry, and so $\overline{m}$ encodes no new information.

Further examples relevant to higher dimensional algebra will be considered in section(7).

We shall now unpack the pseudo monoid part of a monoidal pseudo algebra a little more to facilitate the general operad definition. To this end, let $(A,a,i,m,\alpha,\beta,\lambda,\rho)$ be a monoidal pseudo T-algebra. First, we note that the unit object $i:X\to A$ of the monoidal category $\mathcal{K}(X,A)$ is the composite

$$X\xrightarrow{!}1\xrightarrow{i}A$$

in $\mathcal{K}$, where ! here denotes the unique map into the terminal object. Moreover, given objects x and y of $\mathcal{K}(X, A)$, their tensor product $x \otimes y$ is the composite

$$X \xrightarrow{(x,y)} A^2 \xrightarrow{m} A$$

in $\mathcal{K}$. Note that if $z : Z \to X$, then $(x \otimes y)z = xz \otimes yz$ by the naturality of $\otimes$. Similarly one can express the rest of the pseudo monoid data (β, λ, ρ) representably.

One can write $\bar{i} : aT(i) \to i$ for the 2-cell

$$\begin{array}{ccc} T1 & \xrightarrow{!} & 1 \\ {\scriptstyle T(i)}\downarrow & \overset{\bar{i}}{\Rightarrow} & \downarrow{\scriptstyle i} \\ T(A) & \xrightarrow[a]{} & A \end{array}$$

which provides i's strong T-morphism structure. As for $\overline{m}$, given objects x and y of $\mathcal{K}(X, A)$, we shall write

$$aT(x \otimes y) \xrightarrow{\overline{m}_{x,y}} aT(x) \otimes aT(y)$$

for the composite

$$\begin{array}{ccccccc} T(X) & \xrightarrow{T(x,y)} & T(A \times A) & \xrightarrow{\pi} & T(A) \times T(A) & \xrightarrow{a \times a} & A \times A \\ & & {\scriptstyle T(m)}\downarrow & & \overset{\overline{m}}{\Rightarrow} & & \downarrow{\scriptstyle m} \\ & & T(A) & & \xrightarrow[a]{} & & A \end{array}$$

where π is the canonical comparison, and $\overline{m}$ is m's strong T-morphism structure. When the context is clear we shall drop the subscripts and write

$$\overline{m} : aT(x \otimes y) \to aT(x) \otimes aT(y).$$

In light of this notation, the strong T-morphism axioms for $\overline{m}$, and the T-algebra 2-cell axioms for β, λ and ρ, can be restated as follows.

PROPOSITION 4.3. (1) $\forall x, y \in \mathcal{K}(X, A)$,

$$\begin{array}{ccc} aT(a)T^2(x \otimes y) & \xrightarrow{\alpha T^2(x \otimes y)} & aT(x \otimes y)\mu_{TA} \\ {\scriptstyle aT(\overline{m})}\downarrow & & \\ aT(aT(x) \otimes aT(y)) & & \downarrow{\scriptstyle \overline{m}\mu_{TA}} \\ {\scriptstyle \overline{m}}\downarrow & & \\ aT(a)T^2(x) \otimes aT(a)T^2(y) & \xrightarrow[\alpha T^2(x) \otimes \alpha T^2(y)]{} & aT(x)\mu_{TA} \otimes aT(y)\mu_{TA} \end{array}$$

commutes in $\mathcal{K}(X, A)$.

(2) $\forall x, y \in \mathcal{K}(X, A)$, $\overline{m}_{x,y}\eta_X = 1_{x \otimes y}$.

(3) $\forall x,y,z \in \mathcal{K}(X,A)$,

$$\begin{array}{ccc} aT(x\otimes(y\otimes z)) & \xrightarrow{aT(\beta)} & aT((x\otimes y)\otimes z) \\ {\scriptstyle\overline{m}}\downarrow & & \downarrow{\scriptstyle\overline{m}} \\ aT(x)\otimes aT(y\otimes z) & & aT(x\otimes y)\otimes aT(z) \\ {\scriptstyle\mathrm{id}\otimes\overline{m}}\downarrow & & \downarrow{\scriptstyle\overline{m}} \\ aT(x)\otimes(aT(y)\otimes aT(z)) & \xrightarrow[\beta]{} & (aT(x)\otimes aT(y))\otimes aT(z) \end{array}$$

commutes in $\mathcal{K}(X,A)$.

(4) $\forall x \in \mathcal{K}(X,A)$,

$$\begin{array}{ccc} aT(i\otimes x) & \xrightarrow{\overline{m}} & aT(i)\otimes aT(x) \\ {\scriptstyle aT(\lambda)}\downarrow & & \downarrow{\scriptstyle\overline{i}\otimes\mathrm{id}} \\ aT(x) & \xleftarrow[\lambda]{} & i\otimes aT(x) \end{array} \qquad \begin{array}{ccc} aT(x\otimes i) & \xrightarrow{\overline{m}} & aT(x)\otimes aT(i) \\ {\scriptstyle aT(\rho)}\downarrow & & \downarrow{\scriptstyle\mathrm{id}\otimes\overline{i}} \\ aT(x) & \xleftarrow[\rho]{} & aT(x)\otimes i \end{array}$$

commute in $\mathcal{K}(X,A)$.

When writing diagrams such as those in (4.3), notice that there are situations when objects can be expressed in more than one way. For instance in (4.3)(1) we have $a\mu_A T^2(x\otimes y) = aT(x\otimes y)\mu_{TA}$ by the naturality of μ, although in that diagram we have only recorded $aT(x\otimes y)\mu_{TA}$. In similar situations below, we shall just choose one description of a given object without further comment when there is little risk of confusion.

5. Operads

With the language of monoidal pseudo algebras at our disposal, we are now able to present our general operad definition.

DEFINITION 5.1. Let (T,η,μ) be a 2-monad on a 2-category $\mathcal{K}$ with finite products, and let $(A,a,i,m,\alpha,\beta,\lambda,\rho)$ be a monoidal pseudo T-algebra. A T-*operad* (p,ι,σ) in A consists of a 1-cell $p : T(1)\to A$, together with 2-cells ι and σ

$$\begin{array}{ccc} 1 & \xrightarrow{\eta_1} & T1 \\ & {\scriptstyle i}\searrow \; \overset{\iota}{\Rightarrow} \; \swarrow{\scriptstyle p} & \\ & A & \end{array} \qquad \begin{array}{ccc} T^2 1 & \xrightarrow{\mu_1} & T1 \\ & {\scriptstyle pT(!)\otimes aT(p)}\searrow \; \overset{\sigma}{\Rightarrow} \; \swarrow{\scriptstyle p} & \\ & A & \end{array}$$

such that

$$\begin{array}{ccc} i\otimes p & \xrightarrow{\iota!\otimes\mathrm{id}} & pT(!)\eta_{T1}\otimes p \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle\sigma\eta_{T1}} \\ & & p \end{array} \qquad \begin{array}{ccc} p\otimes aT(i) & \xrightarrow{\mathrm{id}\otimes aT(\iota)} & p\otimes aT(p)T(\eta_1) \\ {\scriptstyle\mathrm{id}\otimes\overline{i}}\downarrow & & \downarrow{\scriptstyle\sigma T(\eta_1)} \\ p\otimes i & \xrightarrow[\rho]{} & p \end{array}$$

commute in $\mathcal{K}(T1, A)$ and

$$\begin{array}{ccc}
pT(!)\otimes aT(pT(!)\otimes aT(p)) & \xrightarrow{\mathrm{id}\otimes\overline{m}} & pT(!)\otimes(aT(p)T^2(!)\otimes aT(a)T^2(p)) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes aT(\sigma)} & & \Big\downarrow{\scriptstyle \beta} \\
pT(!)\otimes aT(p)T(\mu_1) & & (pT(!)\otimes aT(p)T^2(!))\otimes aT(a)T^2(p) \\
\Big\downarrow{\scriptstyle \sigma T(\mu_1)} & & \Big\downarrow{\scriptstyle \sigma T^2(!)\otimes\alpha T^2(p)} \\
p\mu_1 T(\mu_1) & \xleftarrow[\sigma\mu_{T1}]{} & pT(!)\mu_{T1}\otimes aT(p)\mu_{T1}
\end{array}$$

commutes in $\mathcal{K}(T^3 1, A)$.

Definition 5.2. Let (T, η, μ) be a 2-monad on a 2-category $\mathcal{K}$ with finite products, and let $(A, a, i, m, \alpha, \beta, \lambda, \rho)$ be a monoidal pseudo T-algebra. A *morphism* $(p, \iota, \sigma)\to(p', \iota', \sigma')$ of T-operads in A consists of a 2-cell $\phi : p \Longrightarrow p'$ such that

$$\begin{array}{ccc}
i & \xrightarrow{\iota} & p\eta_1 \\
& {\scriptstyle \iota'}\searrow & \Big\downarrow{\scriptstyle \phi\eta_1} \\
& & p'\eta_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
pT(!)\otimes aT(p) & \xrightarrow{\sigma} & p\mu_1 \\
\Big\downarrow{\scriptstyle \phi T(!)\otimes aT(\phi)} & & \Big\downarrow{\scriptstyle \phi\mu_1} \\
p'T(!)\otimes aT(p') & \xrightarrow[\sigma']{} & p'\mu_1
\end{array}$$

commute $\mathcal{K}(1, A)$ and $\mathcal{K}(T^2 1, A)$ respectively.

With the evident composition, one obtains the category $\mathrm{Op}(T, A)$ of T-operads in the monoidal pseudo T-algebra A, and a forgetful functor $\mathrm{Op}(T, A)\to\mathcal{K}(T1, A)$.

Definition 5.3. Let (p, ι, σ) be a T-operad in A. A p-algebra $(x, \overline{x})$ consists of a one-cell $x : 1\to A$ and a 2-cell

$$\begin{array}{ccc}
T1 & \xrightarrow{!} & 1 \\
{\scriptstyle p\otimes aT(x)}\searrow & \overset{\overline{x}}{\Rightarrow} & \swarrow{\scriptstyle x} \\
& A &
\end{array}$$

such that

$$\begin{array}{ccc}
i\otimes x & \xrightarrow{\iota\otimes\mathrm{id}} & p\eta_1\otimes x \\
{\scriptstyle \lambda}\searrow & & \swarrow{\scriptstyle \overline{x}\eta_1} \\
& x &
\end{array}$$

commutes in $\mathcal{K}(1, A)$ and

$$\begin{array}{ccc}
pT(!)\otimes aT(p\otimes aT(x)) & \xrightarrow{\mathrm{id}\otimes\overline{m}} & pT(!)\otimes(aT(p)\otimes aT(a)T^2(x)) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes aT(\overline{x})} & & \Big\downarrow{\scriptstyle \beta} \\
pT(!)\otimes aT(x)T(!) & & (pT(!)\otimes aT(p))\otimes aT(a)T^2(x) \\
\Big\downarrow{\scriptstyle \overline{x}T(!)} & & \Big\downarrow{\scriptstyle \sigma\otimes\alpha T^2(x)} \\
x! & \xleftarrow[\overline{x}\mu_1]{} & p\mu_1\otimes aT(x)\mu_1
\end{array}$$

commutes in $\mathcal{K}(T^2 1, A)$.

Definition 5.4. Let (p, ι, σ) be a T-operad in A. A p-algebra morphism

$$f : (x, \overline{x}) \to (y, \overline{y})$$

consists of $f : x \to y$ in $\mathcal{K}(1, A)$ such that

$$\begin{array}{ccc} p \otimes aT(x) & \xrightarrow{\overline{x}} & x! \\ {\scriptstyle \mathrm{id} \otimes aT(f)} \downarrow & & \downarrow {\scriptstyle f!} \\ p \otimes aT(y) & \xrightarrow[\overline{y}]{} & y! \end{array}$$

commutes in $\mathcal{K}(T1, A)$.

With the evident composition, one obtains the category p-Alg of p-algebras and a forgetful functor $p\text{-Alg} \to \mathcal{K}(1, A)$. We shall now see how the well known operad notions are captured by these general definitions.

Examples 5.5.

(1) For $T = 1_{\mathbf{CAT}}$, a T-operad (p, ι, σ) in A is a monoid M in the monoidal category A. The underlying object of M is picked out by $p : 1 \to A$, and the unit and multiplication are provided by ι and σ respectively. An object of p-Alg is an object of A acted on by M.

(2) For $T = \mathcal{M}$, a T-operad (p, ι, σ) in A is a non-symmetric operad in the braided monoidal category A. In more detail, first recall that $\mathcal{M}(1) = \mathbb{N}$ so $p : \mathcal{M}(1) \to A$ is a sequence of objects $(p_n : n \in \mathbb{N})$ of A. The unit ι amounts to a map $i \to p_1$ in A. As for the substitution, notice that $\mathcal{M}^2(1)$ is a discrete category (ie a set), and an element of $\mathcal{M}^2(1)$ a finite sequence $(n_1, ..., n_k)$ of natural numbers, thus σ amounts to a morphism in A for each such sequence. The map μ_1 takes $(n_1, ..., n_k)$ to its sum $n = \sum_{j=0}^{k} n_i$, and so $p\mu_1 : \mathcal{M}^2(1) \to A$ takes $(n_1, ..., n_k)$ to p_n. The map $\mathcal{M}(!) : \mathcal{M}^2(1) \to \mathcal{M}$ takes $(n_1, ..., n_k)$ to k, and so $p\mathcal{M}(!)$ takes $(n_1, ..., n_k)$ to p_k. The map $\mathcal{M}(p) : \mathcal{M}^2(1) \to \mathcal{M}(A)$ takes $(n_1, ..., n_k)$ to the sequence of objects $(p_{n_1}, ..., p_{n_k})$ of A. Recall that the map $a : \mathcal{M}(A) \to A$ takes a sequence of objects $(x_1, ..., x_k)$ of A to their k-ary tensor product $x_1 \otimes ... \otimes x_k$. Thus the map $a\mathcal{M}(p)$ sends $(n_1, ..., n_k)$ to $p_{n_1} \otimes ... \otimes p_{n_k}$. Thus the map $p\mathcal{M}(!) \otimes a\mathcal{M}(p) : \mathcal{M}^2(1) \to A$ sends $(n_1, ..., n_k)$ to $p_k \otimes (p_{n_1} \otimes ... \otimes p_{n_k})$. Therefore the component of σ corresponding to $(n_1, ..., n_k)$ is a map

$$p_k \otimes (p_{n_1} \otimes ... \otimes p_{n_k}) \longrightarrow p_n$$

in A. The axioms express the usual unit and associativity laws for substitution. Notice how the braiding $\overline{m}$ is necessary to express the associativity of the substitution σ. The category p-Alg is the usual category of algebras for the operad.

(3) For $T = \mathcal{B}$, a T-operad (p, ι, σ) in A is a braided operad in the symmetric monoidal category A. This example differs from the previous one in two respects. The first is that the functoriality of $p : \mathcal{B}(1) \to A$ amounts to equipping each p_n with an action of $\mathbf{Br}_n$, the n-th braid group. The second is that the naturality of σ amounts to the substitution being equivariant with respect to these actions. Similarly, for a p-algebra $(x, \overline{x})$, the naturality of $\overline{x}$ encodes its equivariance as an action on x.

(4) In the same way, for $T = \mathcal{S}$, a T-operad (p, ι, σ) in A is a symmetric operad in the symmetric monoidal category A, with the functorialty of p encoding the symmetric group actions on the p_n, and the naturality of σ encoding the equivariance.

6. Higher operads

In order to understand the motivating examples of this paper, it is necessary to review some of the combinatorial aspects of the globular approach to higher dimensional algebra. For a fuller discussion, see [**Bat98**], [**Web04**], [**Web01**], and [**Lei03**]. Define the category $\mathbb{G}$ to have natural numbers as objects, and a generating subgraph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} 3 \underset{\tau_3}{\overset{\sigma_3}{\rightrightarrows}} \cdots$$

subject to the "cosource/cotarget" equations $\sigma_{n+1}\sigma_n = \tau_{n+1}\sigma_n$ and $\tau_{n+1}\tau_n = \sigma_{n+1}\tau_n$, for every $n \in \mathbb{N}$. The objects of the category $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, are called *globular sets*. Thus, a globular set Z consists of a diagram of sets and functions

$$Z_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} Z_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} Z_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} Z_3 \underset{t_3}{\overset{s_3}{\leftleftarrows}} \cdots$$

so that $s_n s_{n+1} = s_n t_{n+1}$ and $t_n t_{n+1} = t_n s_{n+1}$ for every $n \in \mathbb{N}$. The elements of Z_n are called the n-cells of Z, and the functions s_n and t_n are called source and target functions. Define Z to be *of dimension* n when there are no m-cells for $m > n$. All constructions that we consider below, apply equally well to the category of n-globular sets, where $\mathbb{G}$ is replaced by the full subcategory $\mathbb{G}_{(n)}$ consisting of the natural numbers $\leq n$.

Let Z be a globular set. Recall from [**Str91**] the *solid triangle order* $\blacktriangleleft$ on the elements (of all dimensions) of Z. Define first the relation $x \prec y$ for $x \in Z_n$ iff $x = s_n(y)$ or $t_{n-1}(x) = y$. Then take $\blacktriangleleft$ to be the reflexive-transitive closure of $\prec$. Write $\mathrm{Sol}(Z)$ for the preordered set so obtained. Observe that Sol is the object map of a functor

$$[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{Sol}} \mathbf{PreOrd}$$

where **PreOrd** is the category of preordered sets and order-preserving functions.

Definition 6.1. A *globular cardinal* is a globular set Z such that $\mathrm{Sol}(Z)$ is a non-empty finite linear order.

Denote by Θ_0 the full subcategory of $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ consisting of the globular cardinals. Globular cardinals are the pasting schemes appropriate to the Batanin definition of weak ω-category [**Bat98**], are analysed from the present point of view in [**Web01**]. In particular we have

Proposition 6.2. (1) *Globular cardinals are finite and connected as globular sets.*
(2) *All morphisms in Θ_0 are monic.*
(3) *If X is a globular cardinal, then a retraction $X \to Y$ of globular sets is an isomorphism.*

Write $\mathbf{Tr}_n$ for the set of isomorphism classes of globular cardinals of dimension n. One of the most beautiful ideas in [**Bat98**], is the identification of $\mathbf{Tr}_n$ with n-stage trees, where an n-stage tree T is defined to be a sequence

$$T_n \to ... \to T_0$$

of maps in $\mathbf{\Delta}$, the category of finite ordinals and monotone maps, where $T_0 = 1$. Central to the Batanin approach to higher dimensional algebra is the monad $\mathcal{T}_0$ on $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ whose algebras are strict ω-categories. The underlying functor of this monad can be described as

$$\mathcal{T}_0(X)_n = \sum_{T \in \mathbf{Tr}_n} [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}](T, X),$$

and the multiplication of this monad, which encodes the pasting of globular pasting schemes, can be specified in terms of trees. This monad is *cartesian*, in the sense that the underlying endofunctor preserves pullbacks, and the naturality squares for η and μ are pullback squares. As mentioned in the introduction, one can then regard $\mathcal{T}_0$ as a monad on $[\mathbb{G}^{\mathrm{op}}, \mathbf{SET}]$, and then apply $\mathbf{Cat}$ to obtain the cartesian 2-monad $\mathcal{T}$ on $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}]$.

There is a 2-equivalence between Ps_0-$\mathcal{T}$-Alg and the 2-category MonGlob of monoidal globular categories, monoidal globular functors and monoidal globular natural transformations described in [**Bat98**]. One has 2-functors

$$\mathrm{MonGlob} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Ps}_0\text{-}\mathcal{T}\text{-Alg}$$

Given a monoidal globular category X, by making a choice of bracketting of iterated expressions, one constructs the normal pseudo $\mathcal{T}$-algebra $F(X)$, with the same underlying globular category. On the other hand, given a normal pseudo $\mathcal{T}$-algebra Y, one obtains the monoidal globular category $G(Y)$, with the same underlying globular category, by considering only the nullary and binary operations, and associated coherence data. Using the coherence results of [**Bat98**], one can verify directly that F and G form a 2-equivalence of 2-categories.

Pseudo monoids in MonGlob are particularly easy to describe: to give $X \in [\mathbb{G}^{\mathrm{op}}, \mathbf{Cat}]$ a structure of pseudo monoid in MonGlob, is the same as giving the globular category

$$1 \underset{t}{\overset{s}{\leftleftarrows}} X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} \cdots$$

the structure of a monoidal globular category. Such a structure was called an augmented monoidal globular category in [**Bat98**]. From the discussion of the previous paragraph PsMon(Ps_0-$\mathcal{T}$-Alg) and the 2-category of augmented monoidal globular categories are 2-equivalent.

All of the above definitions and results have n-truncated analogues for $n \in \mathbb{N}$. Namely define $\mathbb{G}_{\leq n}$ to be the full subcategory of $\mathbb{G}$ consisting of the $k \in \mathbb{N}$ such that $k \leq n$ and the theory develops analogously. So the category of n-globular sets is $[{\mathbb{G}_{\leq n}}^{\mathrm{op}}, \mathbf{Set}]$, and $\mathcal{T}_{0,\leq n}$ and $\mathcal{T}_{\leq n}$ are the obvious n-truncated analogues of $\mathcal{T}_0$ and $\mathcal{T}$. We recall some of the main examples from [**Bat98**].

Examples 6.3. (1) There is a 2-functor

$$\mathbf{Span} : \mathbf{CAT} \to [\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}]$$

for which $\mathbf{Span}(\mathcal{E})_n = [(\mathbb{G}/n)^{\mathrm{op}}, \mathcal{E}]$. When $\mathcal{E}$ has pullbacks, there is a canonical monoidal globular structure on $\mathbf{Span}(\mathcal{E})$, and when in addition $\mathcal{E}$ has products, this structure is augmented, with the additional (pseudo monoid) structure being given by pointwise cartesian product in the categories $[(\mathbb{G}/n)^{\mathrm{op}}, \mathcal{E}]$.

(2) A monoidal structure on a category $\mathcal{V}$ amounts to a monoidal 1-globular structure on

$$1 \leftleftarrows \mathcal{V}\ .$$

(3) A braided monoidal structure on a category $\mathcal{V}$ amounts to a monoidal 2-globular structure on

$$1 \leftleftarrows 1 \leftleftarrows \mathcal{V}\ .$$

(4) A symmetric monoidal structure on a category $\mathcal{V}$ amounts to a monoidal n-globular structure on

$$1 \leftleftarrows \cdots \leftleftarrows 1 \leftleftarrows \mathcal{V}$$

where $n \geq 3$.

The **Span** construction was analyzed further in [**Str00**]. In particular, for any small category $\mathcal{C}$ in place of $\mathbb{G}$, there is a 2-adjunction

$$\mathbf{CAT} \underset{\mathbf{Span}_{\mathcal{C}}}{\overset{\mathrm{EL}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} [\mathcal{C}^{\mathrm{op}}, \mathbf{CAT}]$$

where $\mathbf{Span}_{\mathcal{C}}(\mathcal{E})(C) = [(\mathcal{C}/C)^{\mathrm{op}}, \mathcal{E}]$, and $\mathrm{EL}(X)$ is the following category:

- objects are pairs (C, x) where $C \in \mathcal{C}$ and $x \in X(C)$.
- morphisms $(C, x) \to (D, y)$ are pairs (f, α) where $f : D \to C$ in $\mathcal{C}$, and $\alpha : X(f)(x) \to y$ in $X(D)$.
- compositions and identities are inherited in the obvious way from $\mathcal{C}$ and the categories $X(C)$.

When X is discrete, that is, as a functor factors through **SET**, then $\mathrm{EL}(X) = \mathrm{el}(X)^{\mathrm{op}}$, the dual of the usual category of elements of X. If moreover, X is small, that is, factors through **Set**, and $\mathcal{E} = \mathbf{Set}$, then we have

$$\begin{aligned} [\mathcal{C}^{\mathrm{op}}, \mathbf{CAT}](X, \mathbf{Span}_{\mathcal{C}}(\mathbf{Set})) &\cong \mathbf{CAT}(\mathrm{EL}(X), \mathbf{Set}) \\ &= \mathbf{CAT}(\mathrm{el}(X)^{\mathrm{op}}, \mathbf{Set}) \\ &\simeq [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]/X \end{aligned}$$

this last step being a well known equivalence of categories, pseudo natural in X. Let (M, η, μ) be a cartesian monad on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and recall the category M-Coll from [**Kel92**] and [**Web04**], which is the full subcategory of $[[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}], [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]]/M$ consisting of the cartesian natural transformations. Recall also that M-Coll has a strict monoidal structure:

- $\eta : 1 \to M$ is the unit.
- $\phi \otimes \psi$ is the composite

$$ST \xrightarrow{\phi\psi} MM \xrightarrow{\mu} M\ ,$$

and that evaluation at 1 provides an equivalence of categories $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]/M(1) \simeq M$-Coll. That is, given a cartesian monad (M, η, μ) on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, we have

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{CAT}](M(1), \mathbf{Span}_{\mathcal{C}}(\mathbf{Set})) \simeq M\text{-Coll}$$

We now present the higher operads of [**Bat98**].

EXAMPLES 6.4.

1. A $\mathcal{T}$-operad (p, ι, σ) in A amounts to a higher operad in an augmented monoidal globular category A in the sense of [**Bat98**], subject to one caveat. That is, the above definition is in fact more general than that presented in [**Bat98**]. The difference is that in [**Bat98**], further hypotheses on A are required, namely, that A has globular coproducts which are compatible with the monoidal pseudo $\mathcal{T}$-algebra structure of A (see [**Bat98**] for further elaboration). In [**Web**], these hypotheses are seen as another instance of a general notion of distributive monoidal pseudo algebra. These further hypotheses induce a monoidal structure on the category $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}](\mathcal{T}(1), A)$ ([**Bat98**] Theorem 6.1) and operads were defined by Batanin to be monoids in this monoidal category. It can be verified directly that the category of monoids in $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}](\mathcal{T}(1), A)$ is isomorphic to $\mathrm{Op}(T, A)$.
2. For the case $A = \mathbf{Span}(\mathbf{Set})$ of (1), we shall continue to regard $\mathcal{T}$ as a 2-monad on $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}]$, and $\mathcal{T}_0$ as a monad on $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$. Now $\mathcal{T}(1) = \mathcal{T}_0(1)$, and the equivalence
$$[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}](\mathcal{T}_0(1), \mathbf{Span}(\mathbf{Set})) \simeq \mathcal{T}_0\text{-Coll}$$
is in fact a monoidal equivalence. Thus, a $\mathcal{T}$-operad in $\mathbf{Span}(\mathbf{Set})$ amounts to a cartesian monad morphism $\phi_0 : R_0 \to \mathcal{T}_0$, and algebras for this operad amount to algebras for the monad R_0. We shall call such an operad ϕ_0 a *basic* higher operad. There is a basic higher operad whose algebras are weak ω-categories.
3. By (6.3)(4) one can consider $\mathcal{T}_{\leq n}$-operads within symmetric monoidal categories, where $n \geq 3$. Such examples are important for the applications of higher operads to the study of loop spaces, see [**Bat02**] and [**Bat03**].
4. Let $\phi_0 : R_0 \to \mathcal{T}_0$ be a basic higher operad and consider the corresponding cartesian 2-monad morphism $\phi : R \to \mathcal{T}$ obtained by shifting universes and applying the 2-functor **Cat** (take category objects). The induced forgetful 2-functors $\mathcal{T}$-Alg$\to R$-Alg, Ps-$\mathcal{T}$-Alg$\to$Ps-R-Alg, and Ps$_0$-$\mathcal{T}$-Alg$\to$Ps$_0$-R-Alg preserve products, and so in particular ϕ induces a forgetful 2-functor
$$\mathrm{PsMon}(\mathrm{Ps}_0\text{-}\mathcal{T}\text{-Alg}) \to \mathrm{PsMon}(\mathrm{Ps}_0\text{-}R\text{-Alg})$$
ensuring a ready supply of examples of monoidal pseudo R-algebras, that is, every monoidal pseudo $\mathcal{T}$-algebra has an "underlying" monoidal pseudo R-algebra structure. So in particular R-operads, for any basic higher operad ϕ_0, can be defined within any augmented monoidal globular category, but live naturally within a "weaker" environment (monoidal pseudo R-algebras).

7. Symmetric variants of higher operads

In this section, the symmetric analogues of Batanin's higher operads are described. In order to do so, new examples of 2-monads on $[\mathbb{G}^{\mathrm{op}}, \mathbf{CAT}]$ are described, which blend together the 2-monad $\mathcal{T}$, with an appropriate 2-monad $\mathcal{C}$ on $\mathbf{CAT}$.

For instance taking $\mathcal{C}$ to be $\mathcal{B}$, the braided monoidal category 2-monad, the blend alluded to here mixes the combinatorics of trees and pasting diagrams encapsulated by $\mathcal{T}$, with that of braids, and the operad notion corresponding to this new monad is a braided analogue of higher operad. This construction hinges on two things:

(1) The underlying 2-functor of C preserves pullbacks. This is easily observed directly for the examples of interest: $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$.
(2) An alternative description of ω-**Cat**, and more generally R-Alg_0 for a basic higher operad $\phi_0 : R_0 \rightarrow \mathcal{T}_0$ (as in (6.4)(2)), as models for a finite connected limit sketch.

This last point is not particularly surprising, at least for strict ω-categories. Already from Ehresmann (see [**Str86**]) one knows that ω-**Cat** is the category of **Set**-valued models for some sketch.

The concept of a sketch is originally due to Ehresmann and there are by now a number of standard references on the subject: [**MP89**], [**AR94**] and [**BW05**]. We shall now explain how the work of Clemens Berger [**Ber02**] exhibits the algebras of a basic higher operad as the models of a finite connected limit sketch. However as pointed out to the author by Steve Lack, it is possible also to give a more general proof of this sketchability result than that presented here using the results of [**ABLR02**].

Following [**Ber02**] we regard Θ_0 as a Grothendieck site by taking covering families to be jointly epimorphic families of morphisms. Denote by $\mathrm{Shv}(\Theta_0)$ the category of sheaves on the site Θ_0. Let $\phi_0 : R_0 \rightarrow \mathcal{T}_0$ be a basic higher operad, and denote by Θ_R the full subcategory of R_0-Alg whose objects are the R_0-algebras freely generated by the globular cardinals, and write $i_R : \Theta_R \hookrightarrow R_0\text{-Alg}$ for the inclusion. Via the left adjoint $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}] \rightarrow R_0\text{-Alg}$ to the forgetful functor, one can identify Θ_0 as a subcategory of Θ_R. Since R_0 is a finitary monad on $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, R-Alg_0 is locally finitely presentable and so cocomplete. Thus one obtains a "hom-tensor" adjunction

$$[\Theta_R^{\mathrm{op}}, \mathbf{Set}] \underset{\mathcal{N}_R}{\overset{\mathcal{L}_R}{\rightleftarrows}} R_0\text{-Alg} \qquad (\mathcal{L}_R \dashv \mathcal{N}_R)$$

where $\mathcal{L}_R$ is the left kan extension of i_R along the yoneda embedding, and $\mathcal{N}_R(X)(T) = R_0\text{-Alg}(i_R(T), X)$.

Definition 7.1. [**Ber02**] A Θ_R-model is a presheaf $F \in [\Theta_R^{\mathrm{op}}, \mathbf{Set}]$ whose restriction to Θ_0 is a sheaf. Denote by $\mathrm{Mod}(\Theta_R)$ the full subcategory of $[\Theta_R^{\mathrm{op}}, \mathbf{Set}]$ consisting of the Θ_R-models.

Theorem 7.2. [**Ber02**] *For any basic higher operad $\phi_0 : R_0 \rightarrow \mathcal{T}_0$:*

(1) *$\mathcal{N}_R$ is fully faithful.*
(2) *The adjunction $\mathcal{L}_R \dashv \mathcal{N}_R$ restricts to an equivalence* $\mathrm{Mod}(\Theta_R) \simeq R_0\text{-Alg}$.

Note that in general, the fully faithfulness of $\mathcal{N}_R$ is equivalent to the density of i_R.

Examples 7.3. (1) For the basic higher operad $\eta : 1 \rightarrow \mathcal{T}$, (7.2)(2) gives an equivalence $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}] \simeq \mathrm{Shv}(\Theta_0)$. This equivalence can also be seen as a basic consequence of the Giraud theorem from topos theory (see

[**MM91**] pg 589), since Θ_0, which contains the representables, generates $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

(2) $\Theta_{\mathcal{T}}$ is equivalent to the Joyal's category of disks from [**Joy97**] which was denoted by Θ. This equivalence was first exhibited in [**BS00**], and proved in [**Ber02**] and independently in [**MZ01**].

We recall the definition of limit sketch and models thereof.

Definition 7.4. A limit sketch is a 4-tuple $\mathcal{D} = (D, I, F, c)$ where D is a category, I is a set, F is an I-indexed set of functors $F_i : J_i \to D$, and c is an I indexed set of cones $c_i : \Delta(x_i) \Longrightarrow F_i$ (where $\Delta(x_i)$ denotes the functor constant at x_i). We call the set F the *diagrams*, and the set c the *distinguished cones* for the sketch $\mathcal{D}$. Let $\mathcal{E}$ be a category with limits of functors out of J_i. The category $\mathrm{Mod}(\mathcal{D}, \mathcal{E})$, of $\mathcal{E}$-valued models of $\mathcal{D}$, is the full subcategory of $[D, \mathcal{E}]$ consisting of the functors $D \to \mathcal{E}$ which take the cones c_i to limiting cones. Denote by $\mathrm{Mod}(\mathcal{D})$ the category $\mathrm{Mod}(\mathcal{D}, \mathbf{Set})$.

Examples 7.5. (1) It is well known that limit sketches subsume Grothendieck topologies, for, let D be a category, and $\mathcal{J}$ a Grothendieck topology on D. Note that for each sieve $\alpha \hookrightarrow D(-, x) \in \mathcal{J}$, one gets a diagram $\mathrm{el}(\alpha) \to D$ as the discrete fibration corresponding to α, and a cocone c for this diagram with components $c_{(y,f)} = \alpha_y(f)$. In this way one gets a distinguished *cone* in D^{op} for each sieve in $\mathcal{J}$, and so a limit sketch whose underlying category is D^{op}. By definition, **Set**-valued models for this sketch are sheaves for the Grothendieck topology $\mathcal{J}$.

(2) Let $\phi_0 : R_0 \to \mathcal{T}_0$ be a basic higher operad. By the above example, one has a limit sketch whose underlying category is Θ_0^{op} from the Grothendieck topology described above (covering maps are jointly epimorphic families). By composing with the inclusion $\Theta_0 \to \Theta_R$, one has a limit sketch whose underlying category is Θ_R^{op}, and by definition, $\mathrm{Mod}(\Theta_R)$ is the category of **Set**-valued models for this sketch. This sketch does not typically arise from a Grothendieck topology.[3] We shall abuse notation and refer to this sketch as Θ_R, even though the underlying category of this sketch is Θ_R^{op}.

Definition 7.6. A limit sketch $\mathcal{D} = (D, I, F, c)$, is a connected limit sketch when the categories J_i (that is, the domains of the F_i) are connected. $\mathcal{D}$ is a finite limit sketch when the J_i have a finite initial subcategory.

For a finite connected limit sketch, the distinguished limiting cones may be regarded as iterated pullbacks. More precisely, for such a sketch $\mathcal{D}$, one can define $\mathrm{Mod}(\mathcal{D}, \mathcal{E})$ as long as $\mathcal{E}$ has pullbacks, and composition with a pullback preserving functor $\mathcal{E} \to \mathcal{E}'$ induces $\mathrm{Mod}(\mathcal{D}, \mathcal{E}) \to \mathrm{Mod}(\mathcal{D}, \mathcal{E}')$. However, in the general context of (7.5)(1), there is nothing forcing the diagram corresponding to an arbitrary sieve $\alpha \hookrightarrow D(-, x) \in \mathcal{J}$, to be finite or connected. We shall now show that for Θ_0 with the given Grothendieck topology, that this is indeed the case, and so, (7.5)(2) is a finite connected limit sketch.

For a linearly ordered set X, and $x \in X$, we shall write x^+ for the successor of x, which exists as long as x is not the maximum element of X.

[3]For example, $\mathcal{T}_0$-Alg $= \omega$-**Cat** is not a Grothendieck topos, in fact, one can show that it is not even a regular category.

Lemma 7.7. *Let X be a globular cardinal. Regarding $x \in X_n$ as an element of* $\mathrm{Sol}(X)$*, and assuming x is not the maximum element, we have*

$$x^+ = \begin{cases} y & \text{if } s(y) = x \\ t(x) & \text{otherwise} \end{cases}$$

Proof. Suppose that $x = s(y)$ and $x \blacktriangleleft z \blacktriangleleft y$. If $z \neq x$ then we have $x \prec a \blacktriangleleft z$, and so a must be either y or $t(x)$. In the first case, $a = y$, we have $y \blacktriangleleft z \blacktriangleleft y$ and so $y = z$. On the other hand, if $a = t(x)$, note that $t(x) = ts(y) = tt(y)$, and $t(x) \blacktriangleleft z \blacktriangleleft y \blacktriangleleft t(y) \blacktriangleleft tt(y)$, so that $y = z$ also. Thus if $x = s(y)$, we have $x^+ = y$. On the other hand suppose that there is no y such that $x = s(y)$. If there were no $t(x)$, then $x \in X_0$, and X would be the globular set with one 0-cell and no other cells, in which case x is the maximum. Now, suppose $x \blacktriangleleft z \blacktriangleleft t(x)$ and $x \neq z$. Then we have $x \prec a \blacktriangleleft z$ and a is forced to be $t(x)$. Thus, $t(x) \blacktriangleleft z \blacktriangleleft t(x)$ and so $z = t(x)$. Thus if there is no y such that $x = s(y)$, then $x^+ = t(x)$. □

Corollary 7.8. *Let $f : X \to Y$ in Θ_0. Then $\mathrm{Sol}(f)(x^+) = \mathrm{Sol}(f)(x)^+$ for all non-maximal elements x.*

Proof. By (7.7) the successor operation for $\mathrm{Sol}(X)$ is expressed in terms of the sources and targets for X, which are preserved by f since it is a morphism of globular sets. □

Given non-empty finite linear orders X, Y and Z, and successor-preserving maps

$$X \xrightarrow{f} Y \xleftarrow{g} Z \ ,$$

the pullback in **PreOrd** of these maps is a finite linear order. It will simply be formed as the intersection of the images of f and g. Recall that in any category $\mathcal{E}$ with pullbacks and an initial object, arrows f and g as above are said to be *disjoint* when their pullback is the initial object of $\mathcal{E}$.

Proposition 7.9. *Θ_0 has pullbacks of pairs of maps which are non-disjoint in* $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

Proof. Let

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y & \underset{g}{\longrightarrow} & Z \end{array}$$

be a pullback square in $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, f and g non-disjoint, and X, Y and Z globular cardinals. Applying Sol, which preserves pullbacks and initial objects, to this pullback square, exhibits $\mathrm{Sol}(P)$ as a pullback of non-disjoint successor-preserving maps between finite linear orders. Thus, $\mathrm{Sol}(P)$ is a non-empty finite linear order, and so P is a globular cardinal. □

Proposition 7.10. *The limit sketch of (7.5)(2) is a finite connected limit sketch.*

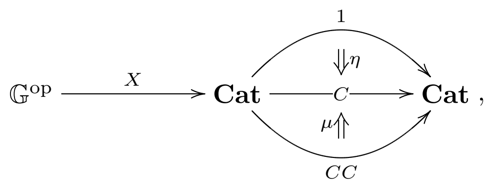
Proof. Let F be a jointly epimorphic family of maps in Θ_0 with codomain X, let $\alpha \hookrightarrow \Theta_0(-, X)$ be the sieve generated by F. We must show that $\mathrm{el}(\alpha)$ is connected and has a finite final subcategory. First note that X is non-empty since it is a globular cardinal, and so F and α are non-empty also. Let f and f' be a pair of maps in F. Then there will be a finite sequence $(f_0, ..., f_n)$ of maps from F, such that all consecutive pairs of maps in the sequence $(f, f_0, ..., f_n, f')$ are non-disjoint, since X is finite and F is a jointly epimorphic family. By (7.9), one can take the joint pullback of the maps $(f, f_0, ..., f_n, f')$ in Θ_0 to exhibit $\mathrm{el}(\alpha)$ as connected. Since all maps in F are monic by (6.2), and X has only finitely many subobjects, $\mathrm{el}(\alpha)$ contains only finitely many maps up to isomorphism in $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]/X$. That is, $\mathrm{el}(\alpha)$ is actually equivalent to a finite category. $\square$

Corollary 7.11. *Let $\phi_0 : R_0 \to \mathcal{T}_0$ be a basic higher operad. Then*

$$R\text{-Alg} \simeq \mathrm{Mod}(\Theta_R, \mathbf{Cat})$$

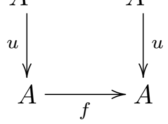
Proof. By (7.10), we can apply the 2-functor $\mathbf{Cat} : \mathbf{PB} \to \mathbf{2CAT}$, which takes category objects (see (2.1)(3)), to the equivalence $\mathrm{Mod}(\Theta_R) \simeq R_0\text{-Alg}$ of (7.2)(2). Clearly, $\mathrm{Mod}(\Theta_R, \mathbf{Cat}) \cong \mathbf{Cat}(\mathrm{Mod}(\Theta_R))$. $\square$

Any 2-monad (C, η, μ) on $\mathbf{Cat}$ may be regarded as a 2-monad $(C_{\mathbb{G}}, \eta_{\mathbb{G}}, \mu_{\mathbb{G}})$ on $[\mathbb{G}^{\mathrm{op}}, \mathbf{Cat}]$ by composition, that is, the components of $\eta_{\mathbb{G}}$ and $\mu_{\mathbb{G}}$ for $X \in [\mathbb{G}^{\mathrm{op}}, \mathbf{Cat}]$ are:

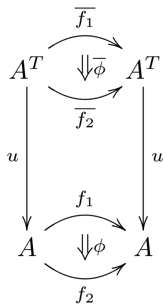


and we shall see that this 2-monad distributes with $\mathcal{T}$ whenever C preserves pullbacks. First, we shall clarify what we mean by a distributive law between 2-monads, since there are various notions that one could use.

Recall that a distributive law between monads S and T, is a natural transformation $\lambda : TS \to ST$, satisfying some axioms, which enable one to define a monad structure on the composite ST. This is done is such a way that algebra structures for ST, amount to compatible S algebra and T algebra structures. In [**Str72**] it was shown that the theory of monad distributive laws can be developed internal to a 2-category. In particular, when the 2-category $\mathcal{K}$ in question has certain weighted limits called Eilenberg-Moore objects, distributive laws $\lambda : TS \to ST$ between monads S and T on $A \in \mathcal{K}$, correspond to liftings of the monad S to the Eilenberg-Moore object A^T (object of T-algebras). In more detail, recall that the Eilenberg Moore object includes a "forgetful one-cell" $u : A^T \to A$. A lifting of $f : A \to A$ is an $\overline{f}$ making



commute. Given liftings $\overline{f_1}$ and $\overline{f_2}$ of f_1 and f_2, and a 2-cell $\phi : f_1 \Longrightarrow f_2$, a lifting of ϕ from $\overline{f_1}$ to $\overline{f_2}$ is a 2-cell $\overline{\phi}$ making



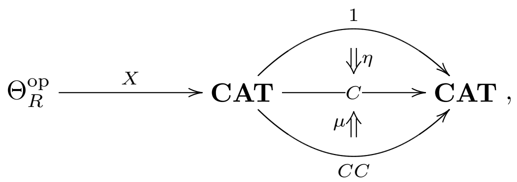
commute. So to give a distributive law $TS \to ST$, is to give a lifting in this sense, of all the data of the monad (S, η, μ) on A, to a monad $(\overline{S}, \overline{\eta}, \overline{\mu})$ on A^T. See [**Str72**] and [**LS02**] for further elaboration. For us, the 2-category $\mathcal{K}$ is that of **CAT**-enriched categories: monads in $\mathcal{K}$ are 2-monads, and the Eilenberg-Moore object of T is the 2-category $T\text{-Alg}_s$ of strict T-algebras, strict algebra morphisms, and algebra 2-cells. When there is a distributive law $TS \to ST$, in this sense between 2-monads S and T, we shall say that S *distributes* with T.

THEOREM 7.12. *Let (C, η, μ) be a 2-monad on* **Cat** *such that C preserves pullbacks, and $\phi_0 : R_0 \to \mathcal{T}_0$ be a basic higher operad. Then the 2-monad $C_{\mathbb{G}}$ distributes with R.*

PROOF. By the theory of distributive laws it suffices to exhibit a lifting of $(C_{\mathbb{G}}, \eta_{\mathbb{G}}, \mu_{\mathbb{G}})$ to $R\text{-Alg}_s$, and by (7.11) we have the equivalence

$$R\text{-Alg}_s \simeq \mathrm{Mod}(\Theta_R, \mathbf{CAT})$$

Since C preserves pullbacks, the 2-monad on $[\Theta_R^{\mathrm{op}}, \mathbf{CAT}]$ with components



restricts to $\mathrm{Mod}(\Theta_R, \mathbf{CAT})$, and is by definition a lifting of $(C_{\mathbb{G}}, \eta_{\mathbb{G}}, \mu_{\mathbb{G}})$. □

EXAMPLE 7.13. Let $\mathcal{E}$ be a category with finite limits, and $\phi_0 : R_0 \to \mathcal{T}_0$ be a basic higher operad. Then by (6.4)(4) and (6.3)(1), $\mathbf{Span}(\mathcal{E})$ has a monoidal pseudo R-algebra structure, with the additional pseudo monoid structure being given dimensionwise by pointwise cartesian product. Thus, this additional pseudo monoid structure may be regarded as a compatible pseudo $\mathcal{S}_{\mathbb{G}}$ algebra structure. In this way, $\mathbf{Span}(\mathcal{E})$ is canonically a pseudo $\mathcal{S}_{\mathbb{G}}R$-algebra. As in (4.2)(4), the pseudo monoid part of a monoidal pseudo $\mathcal{S}_{\mathbb{G}}R$-algebra encodes no new information. Thus, for any basic higher operad $\phi_0 : R_0 \to \mathcal{T}_0$, and category $\mathcal{E}$ with finite limits, $\mathbf{Span}(\mathcal{E})$ is canonically a monoidal pseudo $\mathcal{S}_{\mathbb{G}}R$-algebra. Via the forgetful functors induced by the obvious monad morphisms $\mathcal{M} \to \mathcal{B} \to \mathcal{S}$, $\mathbf{Span}(\mathcal{E})$ may be regarded as a monoidal pseudo algebra also for the monads $\mathcal{M}_{\mathbb{G}}R$ and $\mathcal{B}_{\mathbb{G}}R$.

By (7.12) we can consider $\mathcal{M}_{\mathbb{G}}\mathcal{T}$-operads, $\mathcal{B}_{\mathbb{G}}\mathcal{T}$-operads, and $\mathcal{S}_{\mathbb{G}}\mathcal{T}$-operads – natural higher globular analogues of non-symmetric operads, braided operads and symmetric operads respectively. For that matter, one may replace $\mathcal{T}$ by R, for an arbitrary basic higher operad $\phi_0 : R_0 \to \mathcal{T}_0$. Thus, any higher dimensional categorical structure, which is describable by a basic higher operad, automatically comes equipped with its own analogous notions of non-symmetric, braided, and symmetric operad.

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