In this talk a common generalisation of:

- a simplicial set is the nerve of a category iff it satisfies the Segal condition.

- if $T$ is a finitary monad on Set then there is an equivalence of categories between $T$-Alg and the category of product preserving set valued functors from $\mathcal{L}_T$, the Lawvere theory associated to $T$.

will be presented.

For any functor

$$f : A \to B$$

satisfying certain size conditions, one can consider it's **resolution** which is the functor

$$B(f,1) : B \to \widehat{A}$$

defined by $B(f,1)(b)(a) := B(fa,b)$.

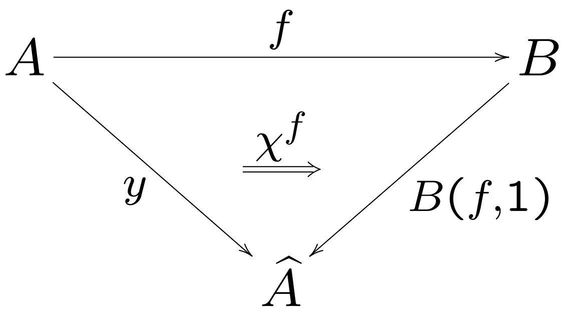
If $f$ is the inclusion of $\Delta$ into Cat, then $B(f,1)$ associates to each category its nerve.

If $f$ is the functor $\Delta \to$Top which sends $[n]$ to the topological $n$-simplex

$$\{x \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_i x_i = 1\}$$

then $B(f,1)$ associates to each space its simplicial resolution.

Resolutions are fundamental to category theory because one has a natural tranformation




whose components

$$(\chi^f{}_{a'})_a : A(a, a') \to B(fa, fa')$$

are the arrow mappings of $f$. The technical aspects of this work were handled by paying attention to the important properties enjoyed by the $\chi^f$'s which have been used to provide 2-categorical axiomatisations of category theory.

**Definition 1.** *A* **monad with arities** $(T, \Theta)$ *consists of a monad* $(T, \eta, \mu)$ *on a cocomplete category* $A$ *together with a fully faithful and dense functor*

$$i_0 : \Theta_0 \to A$$

*such that* $\Theta_0$ *is small, and the functor* $A(i_0, T)$ *preserves the density left extension of* $i_0$*.*

A basic example: take $A$ to be the category of directed graphs, and $T$ the category monad. Then $\Theta_0$ can be taken to consist of the non-empty ordinals.

Another basic example: any finitary monad on Set has arities given by the finite sets.

Each functor can be factored uniquely as an identity on objects followed by a fully faithful. Applying this one defines the category $\Theta_T$ by giving

$$\begin{array}{ccccc} \Theta_0 & \xrightarrow{i_0} & A & \xrightarrow{F} & T\text{-Alg} \\ & {}_{j}\searrow & \| & \nearrow_{i} & \\ & & \Theta_T & & \end{array}$$

where $j$ is an identity on objects functor and $i$ is fully faithful.

**Definition 2.** *The* **nerve functor** *for* $(T, \Theta_0)$ *is*

$$T\text{-Alg}(i, 1) : T\text{-Alg} \to \widehat{\Theta}_T$$

*so for a $T$-algebra $X$ we call* $T$-Alg$(i, X) \in \widehat{\Theta}_T$ *the* **nerve of** $X$*. For $Z \in \widehat{\Theta}_T$ satisfies the* **Segal condition** *when* $\text{res}_j Z$ *is in the image of*

$$A(i_0, 1) : A \to \widehat{\Theta}_0.$$

In the case where $A$ is a presheaf category $\widehat{\mathbb{C}}$ and $\Theta_0$ includes the representables, the Segal condition doesn't look so horribly abstract. For $p \in \Theta_0$, write $\overline{p}$ for the canonical colimit cocone which expresses $p$ as a colimit of representables. Then we have

**Proposition 3.** *$Z \in \widehat{\Theta}_T$ satisfies the Segal condition iff it sends the cocone $j\overline{p}$ to a limit cone.*

**Theorem 4.** *Let $(T, \Theta_0)$ be a monad with arities on $A$.*

1. *The nerve functor is fully faithful.*

2. *$Z \in \widehat{\Theta_T}$ is the nerve of a $T$-algebra iff it satisfies the Segal condition.*

The first step of the proof is to provide isomorphisms

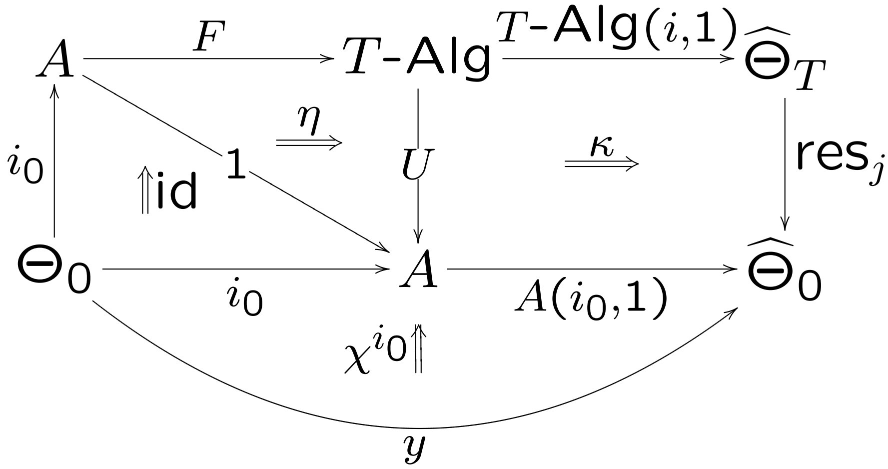
$$\begin{array}{ccc} T\text{-}\mathsf{Alg} & \xrightarrow{T\text{-}\mathsf{Alg}(i,1)} & \widehat{\Theta}_T \\ {\scriptstyle U}\big\downarrow & \overset{\kappa}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathsf{res}_j} \\ A & \xrightarrow[A(i_0,1)]{} & \widehat{\Theta}_0 \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{A(i_0,1)} & \widehat{\Theta}_0 \\ {\scriptstyle F}\big\downarrow & \overset{\psi}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathsf{lan}_j} \\ T\text{-}\mathsf{Alg} & \xrightarrow[T\text{-}\mathsf{Alg}(i,1)]{} & \widehat{\Theta}_T \end{array}$$

which in each case exhibit the right most vertical arrow as a left kan extension. It's obvious that there's an isomorphism $\kappa$ because

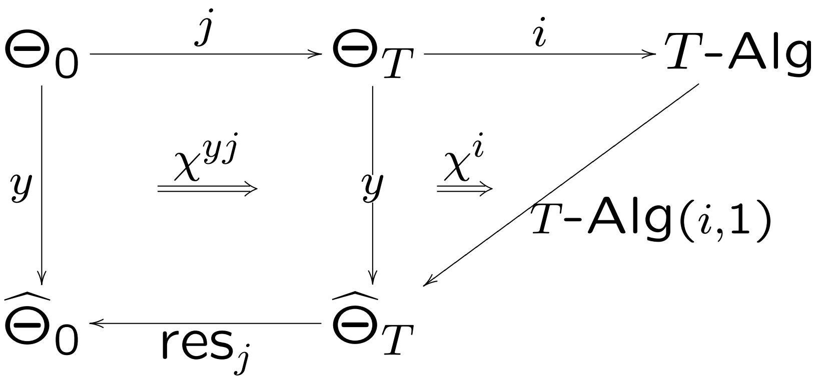
$$A(i_0, U) \cong T\text{-}\mathsf{Alg}(Fi_0, 1) = T\text{-}\mathsf{Alg}(ij, 1)$$

by the adjunction $F \dashv U$.

We can define $\kappa$ as the unique 2-cell such that the composite




is equal to



Writing $\overline{T} := \mathrm{res}_j \mathrm{lan}_j$, by pasting $\kappa$ and $\psi$ one obtains an isomorphism

$$\begin{array}{ccc} A & \xrightarrow{A(i_0,1)} & \widehat{\Theta}_0 \\ {\scriptstyle T}\downarrow & \cong & \downarrow{\scriptstyle \overline{T}} \\ A & \xrightarrow[A(i_0,1)]{} & \widehat{\Theta}_0 \end{array}$$

compatible with the monad structures on $T$ and $\overline{T}$. From such a nice situation one obtains

**Lemma 5.** *There is a bijection between $T$-algebra structures on $a \in A$, and $\overline{T}$ algebra structures on $A(i_0, a)$.*

This expresses the sense in which any monad with arities can be "completed" to a cocontinuous monad on a presheaf category.

Any parametric right adjoint monad $T$ on a presheaf category $\widehat{\mathbb{C}}$ gives an example.

A functor $F : A{\to}B$ is a **parametric right adjoint** when for each $a$ the induced functor

$$A/a \to B/Fa$$

is a right adjoint. When $A$ has a terminal object 1, it's enough to ask this in the case $a{=}1$.

In the case where $B{=}\mathrm{Set}$, such functors are just the coproducts of representables.

If $A$ and $B$ are presheaf categories one has that for $F$ with rank, p.r.a is equivalent to preserving connected limits.

A *monad* is said to be p.r.a when its underlying endofunctor is p.r.a and its unit and multiplication are cartesian.

To give $T$ arities, one should recall another characterisation of p.r.a functors in terms of generic morphisms.

A morphism

$$g : B \to TA$$

is $T$-**generic** when for any $\alpha$, $\beta$, and $\gamma$ making the outside of

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & TX \\ {\scriptstyle f}\downarrow & \overset{T\delta}{\nearrow} & \downarrow{\scriptstyle T\gamma} \\ TA & \xrightarrow[T\beta]{} & TZ \end{array}$$

commute, there is a unique $\delta$ for which $\gamma \circ \delta = \beta$ and $T(\delta) \circ f = \alpha$.

Then the endofunctor $T$ is p.r.a iff each $f : B \to TA$ factors as

$$B \xrightarrow{g} TC \xrightarrow{Th} TA$$

where $g$ is generic.

Now define $p \in \widehat{\mathbb{C}}$ to be an object of $\Theta_0$ when there exists a generic morphism

$$g : \mathbb{C}(-, C) \to Tp$$

When $T$ is the "monoid monad" on Set, $\Theta_0$ consists of the finite sets.

When $T$ is the "category monad" on the category of directed graphs, $\Theta_0$ consists of the finite non-empty ordinals.

When $T$ is the "strict $\omega$ category monad" on the category of globular sets, $\Theta_0$ consists of the globular pasting schemes.

**Proposition 6.** $(T, \Theta_0)$ *is a monad with arities.*

Denote by $\mathbb{M}$ the category with:

- objects: there is an object 0, and for $n \in \mathbb{N}$ an object $(n, 1)$.

- arrows: for each $n \in \mathbb{N}$ there is an arrow $\tau_n : 0 \to (n, 1)$, and for $1 \leq i \leq n$ there are arrows $\sigma_{n,i} : 0 \to (n, 1)$.

An $X \in \widehat{\mathbb{M}}$ is called a *multigraph*. The elements of $X(0)$ are the objects of $X$ and an element of $f \in X(n, 1)$ is said to be a *multi-edge* of *input arity* $n$ and is depicted as

$$f : (a_1, ..., a_n) \to b$$

One can describe a p.r.a monad $T$ on $\widehat{\mathbb{M}}$ whose algebras are symmetric multicategories. The category $\Theta_0$ has isomorphism classes in bijection with planar trees, the objects of $\widehat{\Theta}_T$ are called *dendroidal sets* and our theorem in this case is the characterisation of "nerves of operads" due to Moerdijk and Weiss.