REGULAR PATTERNS, SUBSTITUDES, FEYNMAN CATEGORIES AND OPERADS

MICHAEL BATANIN, JOACHIM KOCK, AND MARK WEBER

Abstract. We show that the regular patterns of Getzler (2009) form a 2-category biequivalent to the 2-category of substitudes of Day and Street (2003), and that the Feynman categories of Kaufmann and Ward (2013) form a 2-category biequivalent to the 2-category of coloured operads (with invertible 2-cells). These biequivalences induce equivalences between the corresponding categories of algebras. There are three main ingredients in establishing these biequivalences. The first is a strictification theorem (exploiting Power's General Coherence Result) which allows to reduce to the case where the structure maps are identity-on-objects functors and strict monoidal. Second, we subsume the Getzler and Kaufmann–Ward hereditary axioms into the notion of Guitart exactness, a general condition ensuring compatibility between certain left Kan extensions and a given monad, in this case the free-symmetric-monoidal-category monad. Finally we set up a biadjunction between substitudes and what we call pinned symmetric monoidal categories, from which the results follow as a consequence of the fact that the hereditary map is precisely the counit of this biadjunction.

Contents

0. Introduction and overview of results

A proliferation of operad-related structures have seen the light in the past decades, such as modular and cyclic operads, properads, and props. Work of many people has sought to develop categorical formalisms covering all these notions on a common footing, and in particular to describe adjunctions induced by the passage from one type of structure to another as a restriction/Kan extension pair [3, 4, 6, 8, 11, 14, 17, 18, 19, 31, 38]. For the line of development of the present work, the work of Costello [11] was especially inspirational:

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in order to construct the modular envelope of a cyclic operad, he presented these notions as symmetric monoidal functors out of certain symmetric monoidal categories of trees and graphs, and arrived at the modular envelope as a left Kan extension corresponding to the inclusion of one symmetric monoidal category into the other. Unfortunately it is not clear from this construction that the resulting functor is even symmetric monoidal. The problem was addressed by Getzler [19] by identifying a condition needed for the construction to work: he introduced the notion of a 'regular pattern' (cf. 0.1 below), which includes a condition formulated in terms of Day convolution, and which guarantees that constructions like Costello's will work. However, his condition is not always easy to verify in practice. Meanwhile, Markl [31], and later Borisov and Manin [8], studied general notions of graph categories, designed with generalised notions of operad in mind, and isolated in particular a certain hereditary condition, which has also been studied by Melliès and Tabareau [32] within a different formalism (cf. 3.2 below). This condition found a comma-category formulation in the recent work of Kaufmann and Ward [25], being the essential axiom in their notion of 'Feynman category', cf. 0.2 below.

Kaufmann and Ward notice that the hereditary axiom is closely related to the Day-convolution Kan extension property of Getzler, and provide an easy-to-check condition under which the envelope construction (and other constructions given by left Kan extensions) work. Their work is the starting point for our investigations.

Another common generalisation of operads and symmetric monoidal categories are the substitudes of Day and Street [14, 15] (in fact considered briefly already by Baez and Dolan [1] under the name C-operad). Their interest came from the study of a nonstandard convolution construction introduced by Bakalov, D'Andrea, and Kac [2]. Substitudes can be also understood as monads in the bicategory of generalised species, introduced by Fiore, Gambino, Hyland and Winskel [17] in 2008.

In the present paper we prove that regular patterns are essentially the same thing as substitudes, and that Feynman categories are essentially the same thing as (coloured) operads. More precisely, we establish biequivalences of 2-categories—this is the best sameness one can hope for, since the involved structures are categorical and hence form 2-categories. For all four notions, a key aspect is their algebras. We show furthermore that under the biequivalences established, the notions of algebras agree. More precisely, if a regular pattern and a substitude correspond to each other under the biequivalence, then their categories of algebras are equivalent. Similarly of course with Feynman categories and operads.

In a broader perspective, our results can be seen as part of a dictionary between two approaches to operad-like structures and their algebras, namely the symmetric-monoidal-category approach and the operadic/multicategorical approach. This dictionary goes back to the origins of operad theory, cf. Chapter 2 of Boardman–Vogt [7]. In fact to establish the results we exploit a third approach, namely that of 2-monads, which goes back to Kelly's paper on clubs [26]. This more abstract approach allows us to pinpoint some essential mechanisms in both approaches. In particular we subsume the Getzler and Kaufmann–Ward hereditary axioms into the 2-categorical notion of Guitart exactness, a general condition ensuring compatibility between certain left Kan extensions and a given 2-monad, in this case the free-symmetric-monoidal-category monad. Since the notion of Guitart exactness has recently proved very useful in operad theory and abstract homotopy theory [4, 20, 24, 30, 38], this interpretation of the axioms of Getzler and Kaufmann–Ward is of independent interest, and we elaborate on it in some detail.

The equivalence between regular patterns and substitudes does not seem to have been foreseen by anybody. The equivalence between Feynman categories and operads may come as a surprise, as Kaufmann and Ward in fact introduced Feynman categories with the intention of providing an 'improvement' over the theory of operads. Part of the structure of Feynman category is an explicit groupoid, which one might think of as a 'groupoid of colours', in contrast to the *set* of colours of an operad. Rather unexpectedly, this groupoid is now revealed to be available already in the usual notion of operad, namely as the groupoid of invertible unary operations. Since the notion of operad is undoubtedly fundamental, the equivalence we establish attests to the importance also of the notion of Feynman category, now to be regarded as a useful alternative viewpoint on operads.

In the present paper, for the sake of focusing on the principal ideas, we work only over the category of sets. For the enriched setting, we refer to Caviglia [10], who independently has established an enriched version of the equivalence between Feynman categories and operads.

We proceed to state our main result, and sketch the ingredients that go into its proof.

For C a category, we denote by $\mathsf{S}C$ the free symmetric monoidal category on C.

0.1. Definition of regular pattern. (Getzler [19]) A *regular pattern* is a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ such that

(1) τ is essentially surjective
(2) the induced functor of presheaves $\tau^* : \widehat{M} \to \widehat{\mathsf{S}C}$ is strong monoidal for the Day convolution tensor product.

0.2. Definition of Feynman category. (Kaufmann–Ward [25]) A *Feynman category* is a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ such that

(1) C is a groupoid
(2) τ induces an equivalence of groupoids $\mathsf{S}C \xrightarrow{\sim} M_{\text{iso}}$
(3) τ induces an equivalence of groupoids $\mathsf{S}(M{\downarrow}C)_{\text{iso}} \xrightarrow{\sim} (M{\downarrow}M)_{\text{iso}}$.

0.3. The hereditary condition. Getzler's definition is staged in the enriched setting. Kaufmann and Ward also give an enriched version called *weak Feynman category* ([25], Definition 4.2 and Remark 4.3) which over ***Set*** reads as follows:

$\tau : \mathsf{S}C \to M$ is an essentially surjective symmetric strong monoidal functor, and the following important *hereditary* condition holds (formulated in more detail in 3.2): For any

$x_1, \ldots, x_m, y_1, \ldots, y_n \in C$, the natural map given by tensoring

$$\sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} M(\bigotimes_{i\in\alpha^{-1}(j)} \tau x_i, \tau y_j) \longrightarrow M(\bigotimes_{i\in\underline{m}} \tau x_i, \bigotimes_{j\in\underline{n}} \tau y_j)$$

is a bijection. (They recognise that this weak notion is 'close' to Getzler's notion of regular pattern but do not prove that it is actually equivalent. In fact this condition does not really play a role in the developments in [25].)

The hereditary condition is natural from a combinatorial viewpoint where it says that every morphism splits into a tensor product of 'connected' morphisms. We shall see (5.13) that in the essentially surjective case it is exactly the condition that the counit for the substitude Hermida adjunction is fully faithful.

0.4. Operads and substitudes. By *operad* we mean coloured symmetric operad in ***Set***. We refer to the colours as objects. The notion of substitude was introduced by Day and Street [14], as a general framework for substitution in the enriched setting. Our substitudes are their symmetric substitudes, cf. also [5], whose appendix constitutes a concise reference for the basic theory of substitudes. A quick definition is this (cf. [15, 6.3]): a *substitude* is an operad equipped with an identity-on-objects operad morphism from a category (regarded as an operad with only unary operations).

We can now state the main theorem:

Theorem. (Cf. Theorem 5.14 and Theorem 5.16.) *There is a biequivalence between the 2-category of substitudes and the 2-category of regular patterns. It restricts to a biequivalence between the 2-category of operads (with invertible 2-cells) and the 2-category of Feynman categories (with invertible 2-cells).*

The biequivalence means that when going back and forth, not an isomorphic object is obtained, but only an equivalent one. This is a question of strictification: one ingredient in the proof is to show that every regular pattern is equivalent to a strict one, and a variant of the main theorem can be stated as a 1-equivalence between these *strict* regular patterns and substitudes. It should be observed that equivalent regular patterns have equivalent algebras (5.19).

We briefly run through the main ingredients of the proof, and outline the contents of the paper.

In Section 1, we show that regular patterns and Feynman categories can be strictified. Both notions concern a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ where $\mathsf{S}C$ is the free symmetric monoidal category on a category C, and in particular is strict. The main result is this:

Proposition. (Cf. Proposition 1.6.) *Every essentially surjective symmetric strong monoidal functor $\mathsf{S}C \to M$, is equivalent to one $\mathsf{S}C \to M'$, for which M' is a symmetric strict monoidal category, and $\mathsf{S}C \to M'$ is strict monoidal and identity-on-objects.*

This is a consequence of Power's coherence result [33], recalled in the appendix. Since the notions of regular pattern and Feynman category are invariant under monoidal equivalence, we may as well work with the strict case, which will facilitate the arguments greatly, and highlight the essential features of the notions, over the subtleties of having coherence isomorphisms everywhere.

The next step, which makes up Section 2, is to put Getzler's condition (2) into the context of Guitart exactness.

0.5. Guitart exactness. Guitart [21] introduced the notion of exact square: they are those squares that pasted on top of a pointwise left Kan extension again gives a pointwise left Kan extension. A morphism of T-algebras for a monad T is exact when the algebra morphism coherence square is exact. We shall need this notion only in the case where T is the free symmetric monoidal category monad on $\boldsymbol{Cat}$: it thus concerns symmetric monoidal functors.

Theorem. (Cf. Theorem 2.11.) *The following are equivalent for a symmetric colax monoidal functor $\tau : S \to M$.*

1. *τ is Guitart exact*
2. *left Kan extension of Yoneda along τ is strong monoidal*
3. *left Kan extension of any strong monoidal functor along τ is again strong monoidal*
4. *$\tau^* : \widehat{M} \to \widehat{S}$ is strong monoidal*
5. *a certain category of factorisations is connected (cf. Lemmas 2.4 and 3.9)*

In the special case of interest to us we thus have

Corollary. *For a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$, axiom (2) of being a regular pattern is equivalent to being exact.*

In Section 3 we analyse the hereditary condition, also shown to be equivalent to a special case of Guitart exactness:

Proposition. (Cf. Proposition 3.3.) *An essentially surjective symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ is exact if and only if it satisfies the hereditary condition.*

Corollary. *A regular pattern is a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ which is essentially surjective and satisfies the hereditary condition.*

Axiom (3) of the notion of Feynman category of Kaufmann and Ward [25], the equivalence of comma categories $\mathsf{S}(M{\downarrow}C)_{\text{iso}} \xrightarrow{\sim} (M{\downarrow}M)_{\text{iso}}$, is of a slightly different flavour to the other related conditions (and in particular, does not seem to carry over to the enriched context). While it is implicit in [25] that this condition is essentially equivalent to the hereditary condition, the relationship is actually involved enough to warrant a detailed proof, which makes up our Section 4.

The outcome is the following result, essentially proved by Kaufmann and Ward [25].

Corollary. *A Feynman category is a special case of a regular pattern, namely such that C is a groupoid and $\mathsf{S}C \to M_{\mathrm{iso}}$ is an equivalence.*

With these two corollaries in place, we can finally establish the promised biequivalences in Section 5. We achieve this by setting up *pinned* variations of the symmetric Hermida adjunction [22] between symmetric monoidal categories and operads:

0.6. Pinned symmetric monoidal categories and pinned operads. A *pinned symmetric monoidal category* is a symmetric monoidal category M equipped with a symmetric strong monoidal functor $\mathsf{S}C \to M$ (where $\mathsf{S}C$ is the free symmetric monoidal category on some category C). Hence regular patterns and Feynman categories are examples of pinned symmetric monoidal categories. Similarly, a *pinned operad* is defined to be an operad equipped with a functor from a category, viewed as an operad with only unary operations. Substitudes are thus pinned operads for which the structure map is identity-on-objects. The latter condition exhibits substitudes as a coreflective subcategory of pinned operads.

0.7. The substitude Hermida adjunction. Our main result will follow readily from the following variation on the Hermida adjunction—actually a biadjunction, which goes between pinned symmetric monoidal categories and substitudes via pinned operads:

$$\boldsymbol{pSMC} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \boldsymbol{pOpd} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \boldsymbol{Subst}. \tag{1}$$

The right adjoint takes a pinned symmetric monoidal category $\mathsf{S}C \to M$ to the substitude

$$C \to \mathrm{End}(M)|C,$$

the endomorphism operad on M, base-changed to C. It is a important feature of substitudes (not enjoyed by operads) that they can be base-changed along functors.

The left adjoint in (1) takes a substitude $C \to P$ to the pinned symmetric monoidal category $\mathsf{S}C \to \mathsf{F}P$, where $\mathsf{F}P$ is the free symmetric monoidal category on P as in the ordinary Hermida adjunction: the objects of $\mathsf{F}P$ are finite sequences of objects in P, and its arrows from sequence $x_1, \ldots, x_m$ to sequence $y_1, \ldots, y_n$ are given by

$$\mathsf{F}P(\mathbf{x}, \mathbf{y}) := \sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} P((x_i)_{i\in\alpha^{-1}(j)}, y_j).$$

The left adjoint is now shown to be fully faithful (5.9). This important feature is not shared by the original Hermida adjunction. Our key result characterises the image of the left adjoint by determining where the counit is invertible:

Proposition. (Cf. Proposition 5.12.) *The counit ε_τ is an equivalence if and only if τ is essentially surjective and the hereditary condition holds.*

Corollary. *The essential image of the left adjoint is the 2-category of regular patterns.*

In particular, this establishes the first part of the Main Theorem:

Theorem. (Cf. Theorem 5.14.) *The left adjoint induces a biequivalence between the* 2*-category of substitudes and the* 2*-category of regular patterns.*

To an operad P one can assign a substitude by taking the canonical groupoid pinning $P_1^{\mathrm{iso}} \to P$. This is not functorial in all 2-cells, only in invertible ones; it is the object part of a fully faithful 2-functor $(\boldsymbol{Opd})^{2\text{-iso}} \to \boldsymbol{Subst}$. We characterise its regular patterns in the image of this 2-functor: they are precisely the Feynman categories. This establishes the second part of the main theorem:

Theorem. (Cf. Theorem 5.16.) *The previous biequivalence induces a biequivalence between the* 2*-category of operads (with invertible* 2*-cells) and the* 2*-category of Feynman categories (with invertible* 2*-cells).*

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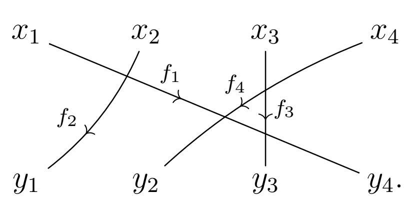
1. Strictification of regular patterns

1.1. The free symmetric monoidal category. The free symmetric monoidal category $\mathsf{S}C$ on a category C has the following explicit description. The objects of $\mathsf{S}C$ are the finite sequences of objects of C. A morphism is of the form

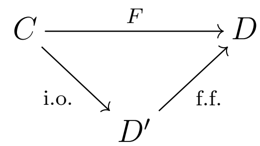
$$(\rho, (f_i)_{i\in\underline{n}}) : (x_i)_{i\in\underline{n}} \longrightarrow (y_i)_{i\in\underline{n}}$$

where $\rho \in \Sigma_n$ is a permutation, and for $i \in \underline{n} = \{1, ..., n\}$, $f_i : x_i \to y_{\rho i}$. Intuitively such a morphism is a permutation labelled by arrows of C, as in

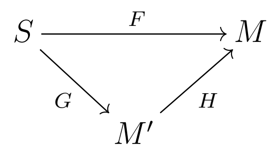


For further details, see the Appendix, where it is explained and exploited that S underlies a 2-monad on $\boldsymbol{Cat}$. It will be important that $\mathsf{S}C$ is actually a symmetric *strict* monoidal category.

1.2. Gabriel factorisation. Given a functor $F : C \to D$, its factorisation into an identity-on-objects followed by a fully faithful functor is referred to as the *Gabriel factorisation* of F:



1.3. Proposition. *Let $F : S \to M$ be a symmetric strong monoidal functor, and assume that S is a symmetric* strict *monoidal category. Then for the Gabriel factorisation*



there is a canonical symmetric strict *monoidal structure on M' for which G is a symmetric strict monoidal functor, and H is canonically symmetric strong monoidal.*

The proof, relegated to the Appendix, exploits the general coherence result of Power [33]. A direct proof is somewhat subtle because the monoidal structure on M' is constructed as a mix of the monoidal structures on S and on M, and it is rather cumbersome to check that the trivial associator defined on M' is actually natural. Instead following Power's approach gives an elegant abstract proof, which exploits the following easily checked facts: (1) the Gabriel factorisation has a 2-dimensional aspect where isomorphisms can always be shifted right in the factorisation; and (2) S preserves this factorisation.

1.4. Pinned symmetric monoidal categories. The following terminology will be justified in Section 5, as part of further pinned notions. A *pinned symmetric monoidal category* is a symmetric monoidal category M equipped with a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ (where C is some category). Pinned symmetric monoidal categories are the objects of a 2-category ***pSMC***. A morphism $(C_1, \tau_1, M_1) \to (C_2, \tau_2, M_2)$ is a triple (F, G, ω) consisting of a functor $F : C_1 \to C_2$, a symmetric strong monoidal functor $G : M_1 \to M_2$, and an invertible monoidal natural transformation $\omega : \tau_2 \circ \mathsf{S}F \simeq G \circ \tau_1$. A 2-cell $(F, G, \omega) \to (F', G', \omega')$ is a pair (α, β), where $\alpha : F \to F'$ is a natural transformation and $\beta : G \to G'$ is a monoidal natural transformation, such that ω pasted with β equals $\mathsf{S}\alpha$ pasted with ω'.

A pinned symmetric monoidal category $\tau : \mathsf{S}C \to M$ is called *strict* when M is a symmetric strict monoidal category and τ is a symmetric strict monoidal functor. A morphism $(F, G, \omega) : (C_1, \tau_1, M_1) \to (C_2, \tau_2, M_2)$ is *strict* when G is a symmetric strict monoidal functor and ω is the identity. The locally full sub-2-category spanned by the strict objects and strict morphisms is denoted ***pSMC***$_{\mathrm{s}}$.

1.5. Regular patterns. (Getzler [19]) A *regular pattern* is a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ such that

(1) τ is essentially surjective
(2) the induced functor of presheaves $\tau^* : \widehat{M} \to \widehat{\mathsf{S}C}$ is strong monoidal for the Day convolution tensor product.

Regular patterns form a 2-category $\boldsymbol{RPat}$, namely the full sub-2-category of $\boldsymbol{pSMC}$ spanned by the regular patterns.

A regular pattern (resp. a morphism of regular patterns) is called *strict* when it is strict as a pinned symmetric monoidal category (resp. a morphism of pinned symmetric monoidal categories). These form thus a full sub-2-category $\boldsymbol{RPat}_{\mathrm{s}} \subset \boldsymbol{pSMC}_{\mathrm{s}}$ and a locally full sub-2-category $\boldsymbol{RPat}_{\mathrm{s}} \subset \boldsymbol{RPat}$.

1.6. Proposition. *The inclusion* 2*-functor* $\boldsymbol{RPat}_{\mathrm{s}} \subset \boldsymbol{RPat}$ *is a biequivalence.*

Proof. If $\tau : \mathsf{S}C \to M$ is a regular pattern, in its Gabriel factorisation (as in Proposition 1.3)

$$\begin{array}{ccc} \mathsf{S}C & \xrightarrow{\tau} & M \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle\phi} & \\ & M' & \end{array}$$

σ is identity-on-objects, and ϕ is a strong monoidal equivalence (say with pseudo-inverse ψ and 2-cell $\omega : \psi\phi \simeq \mathrm{id}_{M'}$). It follows that $\sigma^* : \widehat{M} \to \widehat{\mathsf{S}C}$ is strong monoidal: in any case it is lax monoidal, and since both τ^* and ϕ^* are strong monoidal, also σ^* is strong monoidal. In other words, σ is a strict regular pattern. The triple $(\mathrm{id}_C, \phi, \mathrm{id}) : \sigma \to \tau$ is an equivalence in $\boldsymbol{RPat}$ with pseudo-inverse $(\mathrm{id}_C, \psi, \omega\sigma)$. This shows that τ is equivalent to a strict regular pattern, so the inclusion 2-functor is essentially surjective on objects.

The inclusion 2-functor is locally fully faithful by construction, so it remains to see it is locally essentially surjective, i.e. essentially surjective on morphisms. We need to show, given strict regular patterns σ and σ', that any morphism $(F, G, \omega) : \sigma \to \sigma'$ is equivalent to a strict one $(F, G_{\mathrm{strict}}, \omega_{\mathrm{strict}} = \mathrm{id})$. Since σ is bijective on objects, there is a unique way to define the strict G_{strict} on objects so that ω_{strict} becomes the identity 2-cell. It will be equivalent to G by means of the old ω, which also ensures the functoriality of G_{strict}. It is symmetric strict monoidal since $\mathsf{S}F$ and σ' are. So the inclusion 2-functor is locally essentially surjective, and hence altogether a biequivalence. ∎

1.7. Algebras. Let W be a symmetric monoidal category. An *algebra* for a regular pattern $\tau : \mathsf{S}C \to M$ in W, is a symmetric strong monoidal functor $M \to W$. With morphisms of algebras given by monoidal natural transformations, there is a category $\mathrm{Alg}_\tau(W)$ of algebras of (C, τ, M) in W. A morphism of regular patterns $(C, \tau, M) \to (C', \tau', M')$ induces a functor $\mathrm{Alg}_{\tau'}(W) \to \mathrm{Alg}_\tau(W)$ by precomposition. The following proposition is now clear.

1.8. Proposition. *Equivalent regular patterns have equivalent categories of algebras.* ∎

Together with Proposition 1.6, this justifies emphasising strict regular patterns, as we shall often do. This facilitates extracting equivalent characterisations of condition (2)—Guitart exactness and the hereditary condition, in turn exploited in the final comparison with substitudes.

2. Guitart exactness

In this section and the next we show how the main axioms in the definitions of regular pattern and Feynman category can be subsumed in the theory of Guitart exactness. An important aspect of Guitart exactness is to serve as a criterion for pointwise left Kan extensions to be compatible with algebraic structures. This direction of the theory is developed rather systematically in [38] in the abstract setting of a 2-monad on a 2-category with comma objects. The interesting case for the present purposes is the case of the free-symmetric-monoidal-category monad on ***Cat***, and the issue is then under what circumstances left Kan extensions are symmetric monoidal functors.

We write $\widehat{A} := [A^{\mathrm{op}}, \boldsymbol{Set}]$ for the category of presheaves, and $\mathsf{y}_A : A \to \widehat{A}$ for the Yoneda embedding.

2.1. Exact squares. A 2-cell in ***Cat*** of the form

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{2}$$

(called a *lax square*) is *exact in the sense of Guitart* [21] when for any natural transformation ψ which exhibits l as a pointwise left Kan extension of h along f, the composite

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \\ & {\scriptstyle h}\searrow \; \overset{\psi}{\Rightarrow} \; \swarrow{\scriptstyle l} & \\ & V & \end{array}$$

exhibits lg as a pointwise left Kan extension of hp along q.

Suppose that in this situation A is locally small and f is admissible in the sense [34, 35] that $C(fa, c)$ is small for all $a \in A$ and $c \in C$. One thus has the functor $C(f, 1) : C \to \widehat{A}$ given on objects by $c \mapsto C(f(-), c)$, and the effect on arrows of the functor f can be organised into a natural transformation

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ & {\scriptstyle \mathsf{y}_A}\searrow \; \overset{\chi^f}{\Rightarrow} \; \swarrow{\scriptstyle C(f,1)} & \\ & \widehat{A} & \end{array}$$

which exhibits $C(f,1)$ as a pointwise left Kan extension of y_A along f (see e.g. [35] Example 3.3).

2.2. Lemma. (Cf. Guitart [21].) *A lax square* (2) *in which A and P are small and f is admissible is exact if and only if the composite*

$$\begin{array}{ccccc} P & & \xrightarrow{\quad q \quad} & & B \\ {\scriptstyle p}\downarrow & & \overset{\phi}{\Rightarrow} & & \downarrow{\scriptstyle g} \\ A & & \xrightarrow{\quad f \quad} & & C \\ & {\scriptstyle \mathsf{y}_A}\searrow & \overset{\chi^f}{\Rightarrow} & \swarrow{\scriptstyle C(f,1)} & \\ & & \widehat{A} & & \end{array} \tag{3}$$

exhibits $C(f,1)\circ g$ as a pointwise left Kan extension of $\mathsf{y}_A \circ p$ along q. ∎

When $f = \mathsf{y}_A$, the 2-cell χ^f is the identity, and we get the following.

2.3. Corollary. *If P and A are small and*

$$\begin{array}{ccc} P & \xrightarrow{\quad q \quad} & B \\ {\scriptstyle p}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\quad \mathsf{y}_A \quad]{} & \widehat{A} \end{array}$$

exhibits g as a pointwise left Kan extension of $\mathsf{y}_A \circ p$ along q, then ϕ is exact. ∎

Exact squares can be recognised in elementary terms in the following way. First given $a \in A$, $b \in B$ and $\gamma : fa \to gb$ we denote by $\mathrm{Fact}_\phi(a,\gamma,b)$ the following category. Its objects are triples (α, x, β) where $x \in P$, $\alpha : a \to px$ and $\beta : qx \to b$, such that $g(\beta)\phi_x f(\alpha) = \gamma$. Informally, such an object is a 'factorisation of γ through ϕ'. A morphism $(\alpha_1, x_1, \beta_1) \to (\alpha_2, x_2, \beta_2)$ of such is an arrow $\delta : x_1 \to x_2$ such that $p(\delta)\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 q(\delta)$. Identities and compositions are inherited from P.

2.4. Lemma. [21] *A lax square (2) in* ***Cat*** *is exact if and only if for all $a \in A$, $b \in B$, and $\gamma : fa \to gb$, the category $\mathrm{Fact}_\phi(a,\gamma,b)$ defined above is connected.* ∎

2.5. Exact monoidal functors. In the usual nullary-binary way of writing tensor products in monoidal categories, a *symmetric colax monoidal functor* $f : A \to B$ has coherence morphisms of the form

$$\overline{f}_0 : fI \longrightarrow I \qquad \overline{f}_{X,Y} : f(X \otimes Y) \longrightarrow fX \otimes fY$$

(in which I denotes the unit of either A or B) which are required to satisfy axioms that express compatibility with the coherences which define the symmetric monoidal structures on A and B. Equivalently one can regard a symmetric colax monoidal structure on f as comprising coherence morphisms

$$\overline{f}_{X_1,\ldots,X_n} : f(X_1 \otimes \cdots \otimes X_n) \longrightarrow f(X_1) \otimes \cdots \otimes f(X_n)$$

for each sequence $(X_1, ..., X_n)$ of objects of A, whose naturality is expressed by the fact that they are the components of a natural transformation

$$\begin{array}{ccc} \mathsf{S}A & \xrightarrow{\mathsf{S}(f)} & \mathsf{S}B \\ {\scriptstyle\bigotimes}\downarrow & \overset{\overline{f}}{\Longrightarrow} & \downarrow{\scriptstyle\bigotimes} \\ A & \xrightarrow[f]{} & B. \end{array}$$

We say that f is *exact* when this square is an exact square in the sense discussed above. In terms of the 2-monad S, $(f, \overline{f})$ is a colax morphism of pseudo algebras, and in [38], the theory of exact colax morphisms of algebras is developed at the general level of a 2-monad on a 2-category with comma objects.

The following lemma is key to the interest in exactness in the present context. The result is a special case of Theorem 2.4.4 of [38]. A similar result is obtained in the context of proarrow equipments in [32] and in a double categorical setting in [29].

2.6. Lemma. [38] *Let $f : A \to B$ be an exact symmetric colax monoidal functor. Then for any lax symmetric monoidal functor $g : A \to C$ (with C assumed algebraically cocomplete), the pointwise left Kan extension* $\mathrm{lan}_f\, g$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g}\searrow \;\Rightarrow\; \swarrow{\scriptstyle \mathrm{lan}_f\, g} & \\ & C & \end{array}$$

is again naturally lax symmetric monoidal. Furthermore, if g is strong, then so is $\mathrm{lan}_f\, g$.

■

The condition that C is algebraically cocomplete (with respect to f) means first of all that it has enough colimits for the left Kan extension in question to exist, and second, that these colimits are preserved by the tensor product in each variable. More formally, whenever ψ exhibits h as a pointwise left Kan extension of g along f as on the left, then the composite on the right

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g}\searrow \overset{\psi}{\Rightarrow} \swarrow{\scriptstyle h} & \\ & C & \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{S}A & \xrightarrow{\mathsf{S}f} & \mathsf{S}B \\ & {\scriptstyle \mathsf{S}g}\searrow \overset{\mathsf{S}\psi}{\Rightarrow} \swarrow{\scriptstyle \mathsf{S}h} & \\ & \mathsf{S}C & \\ & \downarrow{\scriptstyle\bigotimes} & \\ & C & \end{array}$$

exhibits $\bigotimes \circ \mathsf{S}h$ as a pointwise left Kan extension of $\bigotimes \circ \mathsf{S}g$ along $\mathsf{S}f$.

2.7. Day convolution tensor product. It is well known that the symmetric monoidal structure on A (assumed to be small) extends essentially uniquely to one on $\widehat{A}$, for which the tensor product is cocontinuous in each variable. This tensor product $\ast : \mathsf{S}\widehat{A} \to \widehat{A}$ is called *Day convolution* [13]. It is folklore that the Day convolution tensor product can also be characterised as a pointwise left Kan extension as in the following result, which is nothing more than a translation of the universal property of convolution as expressed by Im and Kelly [23], in these terms.

2.8. Proposition. [23] *For A a small symmetric monoidal category, the Day convolution tensor product $\ast$ on $\widehat{A}$ can be characterised as the pointwise left Kan extension of $\mathsf{y}_A \circ \bigotimes$ along Sy_A,*

$$\begin{array}{ccc} \mathsf{S}A & \xrightarrow{\mathsf{Sy}_A} & \mathsf{S}\widehat{A} \\ {\scriptstyle\otimes}\downarrow & \overset{\bar{\mathsf{y}}_A}{\Longrightarrow} & \downarrow{\scriptstyle\ast} \\ A & \xrightarrow[\mathsf{y}_A]{} & \widehat{A}. \end{array}$$

Furthermore, this square (invertible since Sy_A is fully faithful) constitutes the coherence data making y_A a symmetric strong monoidal functor. Finally, the following universal property holds (usually taken as the defining property of the Day convolution tensor product): For any cocomplete symmetric monoidal category X, composition with y_A gives equivalences of categories

$$\mathrm{Cocts}\boldsymbol{SMC}_c(\widehat{A}, X) \simeq \boldsymbol{SMC}_c(A, X) \qquad \mathrm{Cocts}\boldsymbol{SMC}(\widehat{A}, X) \simeq \boldsymbol{SMC}(A, X).$$

Here $\boldsymbol{SMC}$ is the 2-category of symmetric monoidal categories with symmetric strong monoidal functors, while $\boldsymbol{SMC}_c$ has also symmetric colax monoidal functors. The prefixes Cocts indicate the full subcategories spanned by cocomplete symmetric monoidal categories whose tensor product preserves colimits in both variables.

Proof. For any category C, we denote by $\mathsf{M}C$ the free (strict) monoidal category on C. Explicitly $\mathsf{M}C$ is the subcategory of $\mathsf{S}C$ containing all the objects, but just the morphisms whose underlying permutation is an identity. The inclusions $i_C : \mathsf{M}C \to \mathsf{S}C$ are the components of a 2-natural transformation $i : \mathsf{M} \to \mathsf{S}$ which by the results of [36], conforms to the hypotheses of Proposition 4.6.2 of [38]. Thus for any functor $f : C \to D$, the corresponding naturality square of i on the left

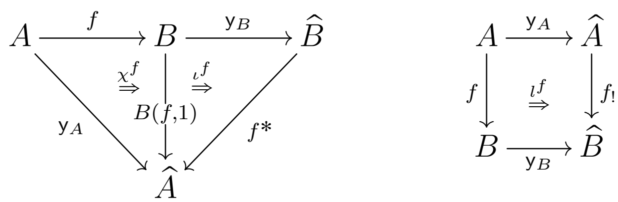
$$\begin{array}{ccc} \mathsf{M}C & \xrightarrow{\mathsf{M}f} & \mathsf{M}D \\ {\scriptstyle i_C}\downarrow & & \downarrow{\scriptstyle i_D} \\ \mathsf{S}C & \xrightarrow[\mathsf{S}f]{} & \mathsf{S}D \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{M}A & \xrightarrow{\mathsf{My}_A} & \mathsf{M}\widehat{A} \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_{\widehat{A}}} \\ \mathsf{S}A & \xrightarrow{\mathsf{Sy}_A} & \mathsf{S}\widehat{A} \\ {\scriptstyle\otimes}\downarrow & \overset{\bar{\mathsf{y}}_A}{\Longrightarrow} & \downarrow{\scriptstyle\ast} \\ A & \xrightarrow[\mathsf{y}_A]{} & \widehat{A} \end{array}$$

is exact, and so the composite square on the right exhibits $\ast \circ i_{\widehat{A}}$ as a pointwise left Kan extension, and this functor has the same object map as $\ast$. Computing the left Kan extension on the left in the previous display as a coend in the usual way, one recovers the usual formula for the Day tensor product. Thus the result follows from [23]. ∎

With Corollary 2.3, we arrive at the following.

2.9. Corollary. *$\mathsf{y}_A : (A, \otimes) \to (\widehat{A}, *)$ is exact.* ∎

2.10. Generalities. For any functor $f : A \to B$ between small categories, we have the 2-cells



exhibiting $B(f,1)$ as the pointwise left Kan extension of y_A along f, and f^* (restriction along f) as the pointwise left Kan extension of $B(f,1)$ along y_B. Finally, l^f exhibits $f_!$ (the left adjoint to f^*) as the pointwise left Kan extension of $\mathsf{y}_B \circ f$ along y_A. Note that l^f is an exact square by Corollary 2.3, and that both ι^f and l^f are invertible, since y_B and y_A are fully faithful.

Returning to our situation of a symmetric colax monoidal functor between small symmetric monoidal categories $(f, \overline{f}) : A \to B$, by Lemma 2.6 $f_!$ gets a symmetric colax monoidal structure from that of $\mathsf{y}_B \circ f$, since y_A is exact by Proposition 2.8. The colax coherence datum $\overline{f}_!$ (which we don't make explicit here, and which is invertible if and only if $\overline{f}$ is) induces, by taking mates via $f_! \dashv f^*$, the coherence 2-cell $\overline{f}^*$ making f^* a lax monoidal functor. Moreover $B(f,1)$ gets a unique monoidal structure making ι^f an invertible monoidal natural transformation. In the context just described we have the following alternative characterisations of exactness of the symmetric colax monoidal functor $(f, \overline{f})$.

2.11. Theorem. *The following statements are equivalent for a colax symmetric monoidal functor $f : A \to B$ (assuming A small and f admissible).*

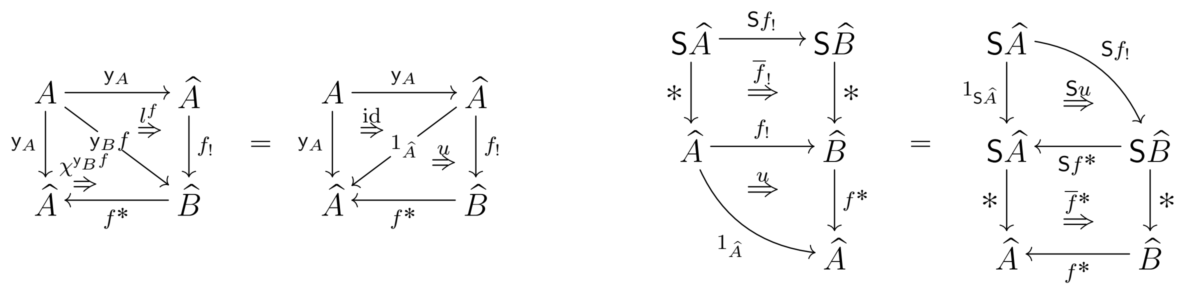
1. *f is exact.*
2. *For any algebraically cocomplete symmetric monoidal category X and any symmetric strong monoidal functor $g : A \to X$, the pointwise left Kan extension of g along f is symmetric strong monoidal.*
3. *$B(f,1) : B \to \widehat{A}$ is symmetric strong monoidal.*
4. *$f^* : \widehat{B} \to \widehat{A}$ is symmetric strong monoidal.*

Proof. (1) $\Longrightarrow$ (2): The assumptions imply that the left Kan extension exists. The statement now follows from Lemma 2.6.

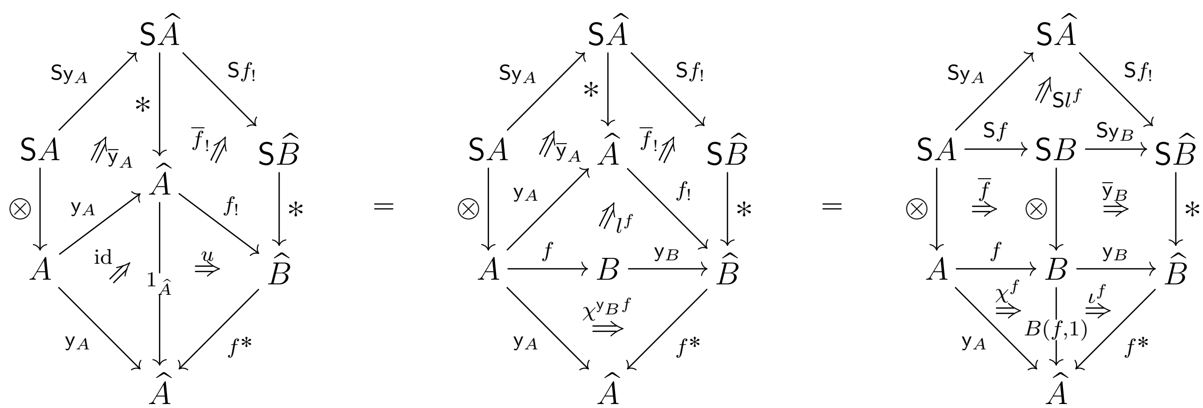
(2) $\Longrightarrow$ (3): By Proposition 2.8, y_A is strong monoidal. Since χ^f exhibits $B(f,1)$ as a pointwise left Kan extension of y_A along f, we conclude by the assumption (2) that $B(f,1)$ is strong monoidal.

(3) $\implies$ (4): By Corollary 2.9, y_B is exact, and $B(f,1)$ is strong monoidal by assumption. But ι^f exhibits f^* as the pointwise left Kan extension of $B(f,1)$ along y_B, so again by Lemma 2.6 we conclude that f^* is strong monoidal.

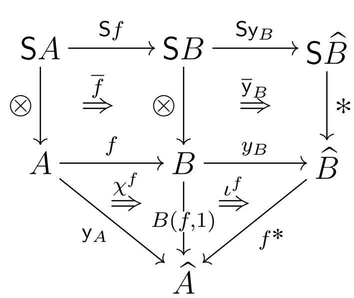
(4) $\implies$ (1): From [34, 35] the unit u of the adjunction $f_! \dashv f^*$ is the unique 2-cell satisfying the equation on the left



and $\overline{f}_!$ and $\overline{f}^*$ determine each other uniquely by the equation on the right. Being the unit of an adjunction, $\mathsf{S}u$ is an absolute pointwise left Kan extension, and since $\overline{f}^*$ is assumed to be invertible (4), the common composite of the equation on the right in the previous display exhibits $f^* \circ \circledast_B$ as the pointwise left Kan extension of $\circledast_A$ along $\mathsf{S}f_!$. Now paste on the left with $\overline{\mathsf{y}}_A$ which is a pointwise left Kan extension by Proposition 2.8. The resulting pointwise left Kan extension can be rewritten as follows:



and since $\mathsf{S}l^f$ is invertible, we conclude that already



exhibits $f^* \circ \circledast_B$ as a pointwise left Kan extension of $\mathsf{y}_A \circ \bigotimes_A$ along $\mathsf{S}\mathsf{y}_B \circ \mathsf{S}f$. But already ι^f is a pointwise left Kan extension, and $\overline{\mathsf{y}}_B$ is an exact square, so the whole right-hand part of the diagram is a pointwise left Kan extension. Since furthermore Sy_B is fully faithful, we can cancel that right-hand part away (e.g. by [27], Theorem 4.47), so in conclusion also

$$\begin{array}{ccc} \mathsf{S}A & \xrightarrow{\mathsf{S}f} & \mathsf{S}B \\ {\scriptstyle\otimes}\downarrow & \overset{\overline{f}}{\Rightarrow} & \downarrow{\scriptstyle\otimes} \\ A & \xrightarrow{f} & B \\ & \overset{\chi^f}{\Rightarrow} & \\ {\scriptstyle \mathsf{y}_A}\searrow & & \swarrow{\scriptstyle B(f,1)} \\ & \widehat{A} & \end{array},$$

is a pointwise left Kan extension. It now follows from Lemma 2.2 that f is exact.

Note that the implication (4) $\implies$ (2) was established already by Getzler [19], and in fact can be extracted from Bunge–Funk [9], Proposition 1.5, as pointed out by the anonymous referee. ■

2.12. Corollary. *For a symmetric monoidal functor $\tau : \mathsf{S}C \to M$, axiom (2) of being a regular pattern is equivalent to being exact.* ■

2.13. Morphisms of regular patterns. Recall (from 1.5) that a morphism of regular patterns is a diagram of symmetric strong monoidal functors

$$\begin{array}{ccc} \mathsf{S}C_1 & \xrightarrow{\tau_1} & M_1 \\ {\scriptstyle \mathsf{S}f}\downarrow & \overset{\omega}{\simeq} & \downarrow{\scriptstyle g} \\ \mathsf{S}C_2 & \xrightarrow[\tau_2]{} & M_2, \end{array}$$

where ω is an invertible monoidal natural transformation.

2.14. Proposition. *Every such $g : M_1 \to M_2$ is exact.*

Proof. The free functor $\mathsf{S}C_1 \to \mathsf{S}C_2$ is exact by Corollary 4.6.6 of [38]. The two functors τ_1 and τ_2 are exact by assumption, and τ_1 is furthermore bijective on objects. It now follows from Lemma 2.15 that g is exact. ■

2.15. Lemma. *Given a commutative triangle of symmetric strong monoidal functors*

$$\begin{array}{ccc} S & \xrightarrow{f} & T, \\ {\scriptstyle u}\searrow & & \nearrow{\scriptstyle g} \\ & S' & \end{array}$$

if f is exact, and u is exact and bijective on objects, then g is exact.

Proof. By Theorem 2.11 it is enough to check that g^* is strong monoidal. Consider the corresponding triangle of pullback functors:

$$\begin{array}{ccc} \widehat{S} & \xleftarrow{\;f^*\;} & \widehat{T} \\ {\scriptstyle u^*}\nwarrow & & \swarrow{\scriptstyle g^*} \\ & \widehat{S}' & \end{array}$$

All three functors are lax monoidal; f^* and u^* are strong monoidal because of exactness. Furthermore u^* is monadic since u is bijective on objects, and so u^* is conservative. The monoidal coherences of f^* are invertible; but these are obtained by applying u^* to the lax coherence of g^*. Since u^* is conservative we can therefore conclude that already the coherences for g^* must be invertible. ∎

3. The hereditary condition and exactness

In this section we analyse the hereditary condition of Kaufmann and Ward [25] and relate it to Guitart exactness in Proposition 3.3. In Section 5 we shall see that the hereditary condition is one of two conditions characterising substitudes among pinned monoidal categories (Proposition 5.12).

3.1. Permutation-monotone factorisation. As in Section 1 for $n \in \mathbb{N}$, we denote by $\underline{n}$ the linearly-ordered set $\{1, ..., n\}$. Any function $\alpha : \underline{m} \to \underline{n}$ factors uniquely as

$$\alpha = \lambda_\alpha \circ \sigma_\alpha,$$

where $\sigma_\alpha : \underline{m} \xrightarrow{\sim} \underline{m}$ is a permutation that is monotone on the fibre $\alpha^{-1}(j)$ for each $j \in \underline{n}$, and $\lambda_\alpha : \underline{m} \to \underline{n}$ is monotone[1]. With reference to $\alpha : \underline{m} \to \underline{n}$, if $(x_1, \ldots, x_m) = (x_i)_{i\in\underline{m}}$ is a sequence of objects, we denote by $(x_i)_{\alpha i=j}$ the subsequence consisting of those entries whose index maps to j. The order is the induced order on the subset $\alpha^{-1}(j) \subset \underline{m}$.

3.2. The hereditary condition. A symmetric colax monoidal functor $\tau : \mathsf{S}C \to M$ satisfies the *hereditary condition* when for all pairs of sequences $(x_i)_{i\in\underline{m}}$ and $(y_j)_{j\in\underline{n}}$ of objects of C, the function

$$\mathbf{h}_{\tau,x,y} : \sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} M(\tau(x_i)_{\alpha i=j}, \tau y_j) \longrightarrow M(\tau(x_i)_{i\in\underline{m}}, \textstyle\bigotimes_{j\in\underline{n}} \tau y_j)$$

which sends $(\alpha, (g_j)_{j\in\underline{n}})$ to the composite

$$\tau(x_i)_{i\in\underline{m}} \xrightarrow{\tau\sigma_\alpha} \tau(x_{\sigma_\alpha^{-1} i})_{i\in\underline{m}} \xrightarrow{\overline{\tau}} \textstyle\bigotimes_{j\in\underline{n}} \tau(x_i)_{\alpha i=j} \xrightarrow{\bigotimes_j g_j} \bigotimes_{j\in\underline{n}} \tau y_j$$

[1]This factorisation is not part of a factorisation system, but it is nevertheless very useful.

in M, is a bijection. (Note that $\bigotimes_{j\in\underline{n}}\tau(x_{\sigma_\alpha^{-1}i})_{\lambda_\alpha i=j} = \bigotimes_{j\in\underline{n}}\tau(x_i)_{\alpha i=j}$.) Note that the summation is taken over *arbitrary* functions $\alpha : \underline{m} \to \underline{n}$, not just monotone ones.

In the case where τ is strict (i.e. when $\overline{\tau}$ is the identity) one has $\tau(x_i)_{i\in\underline{m}} = \bigotimes_{i\in\underline{m}}\tau x_i$, and it may be more convenient to write $\mathbf{h}_{\tau,x,y}$ as the function

$$\sum_{\alpha:\underline{m}\to\underline{n}}\prod_{j\in\underline{n}} M(\textstyle\bigotimes_{\alpha i=j}\tau x_i, \tau y_j) \longrightarrow M(\bigotimes_{i\in\underline{m}}\tau x_i, \bigotimes_{j\in\underline{n}}\tau y_j)$$

which sends $(\alpha, (g_j)_{j\in\underline{n}})$ to the composite

$$\bigotimes_{i\in\underline{m}}\tau x_i \xrightarrow{\sigma} \bigotimes_{i\in\underline{m}}\tau x_{\sigma^{-1}i} = \bigotimes_{j\in\underline{n}}\bigotimes_{\alpha i=j}\tau x_i \xrightarrow{\otimes_j g_j} \bigotimes_{j\in\underline{n}}\tau y_j.$$

In less formal terms, the hereditary condition says that every morphism f of M as on the right

$$g_j : \textstyle\bigotimes_{\alpha i=j}\tau x_i \longrightarrow \tau y_j \qquad\qquad f : \bigotimes_{i\in\underline{m}}\tau x_i \longrightarrow \bigotimes_{j\in\underline{n}}\tau y_j$$

can be uniquely decomposed as a tensor product of morphisms g_j as on the left, modulo some symmetry coherence isomorphisms in M. A useful slogan for this is: 'many-to-many maps decompose uniquely as a tensor product of many-to-one maps'; which expresses the operadic nature of this condition. The hereditary condition has been discovered independently by various people in different guises. While Kaufmann and Ward got it from Markl [31] via Borisov and Manin [8], it is also equivalent to the (operad case of the) 'operadicity' condition of Melliès and Tabareau [32, §3.2].

The main result of this section is

3.3. Proposition. *Let $\tau : \mathsf{S}C \to M$ be a symmetric colax monoidal functor.*

(1) *If τ is exact then it satisfies the hereditary condition.*
(2) *If τ is essentially surjective, strong monoidal and satisfies the hereditary condition, then τ is exact.*

and its proof occupies the rest of this section. For (1), we shall first show that the sum-over-functions formula arises from the Day convolution product, and second that the hereditary maps are special cases of the components of a canonical 2-cell associated to τ. For (2), we shall invoke a classical criterion for exactness in terms of a category of factorisations, going back to Guitart himself [21] in some form, and analysed in more detail in [38].

3.4. Sums over functions from convolution for free symmetric monoidal categories. Our discussion begins by identifying how sums over functions, as in the domains of the hereditary condition maps $\mathbf{h}_{\tau,x,y}$, arise categorically. For a small category C we define the functor

$$\ast : \mathsf{S}(\widehat{\mathsf{S}C}) \longrightarrow \widehat{\mathsf{S}C}$$

to be given on objects as

$$(\mathop{\ast}_{j\in\underline{n}} F_j)(x_i)_{i\in\underline{m}} \;=\; \sum_{\alpha:\underline{m}\to\underline{n}}\prod_{j\in\underline{n}} F_j(x_i)_{\alpha i=j} \tag{4}$$

where the sum is taken over all functions $\underline{m} \to \underline{n}$. We shall now see, as the notation chosen indicates, that $\circledast$ is the Day convolution tensor product.

Let us first exhibit the functoriality of $\circledast_j F_j$ in $(x_i)_{i\in\underline{m}}$ (that is, verify that the assignment in (4) really defines a presheaf on $\mathsf{S}C$). Given $(F_j)_{j\in\underline{n}}$ in $\mathsf{S}(\widehat{\mathsf{S}C})$ and a morphism $(\sigma,(f_i)_i) : (x_i)_{i\in\underline{m}} \to (x'_i)_{i\in\underline{m}}$ in $\mathsf{S}C$, note that the permutation $\sigma \in \Sigma_m$ restricts to $\sigma_j : \alpha^{-1}(j) \to (\alpha\sigma^{-1})^{-1}(j)$ for $j \in \underline{n}$, and so we get $(\sigma_j,(f_i)_{\alpha i=j}) : (x_i)_{\alpha i=j} \to (x'_i)_{\alpha i=j}$ in $\mathsf{S}C$ for each $j \in \underline{n}$. Thus we define $\underset{j\in\underline{n}}{\circledast} F_j(\sigma,(f_i)_i)$ as the unique function such that the square

$$\begin{array}{ccc}
\prod_{j\in\underline{n}} F_j(x_i)_{\alpha i=j} & \xrightarrow{k_\alpha} & (\underset{j\in\underline{n}}{\circledast} F_j)(x_i)_{i\in\underline{m}} \\
{\scriptstyle \prod_j(\sigma_j,(f_i)_{\alpha i=j})}\downarrow & & \downarrow{\scriptstyle \circledast_j F_j(\sigma,(f_i)_i)} \\
\prod_{j\in\underline{n}} F_j(x'_i)_{\alpha\sigma^{-1}i=j} & \xrightarrow[k_{\alpha\sigma^{-1}}]{} & (\underset{j\in\underline{n}}{\circledast} F_j)(x'_i)_{i\in\underline{m}}
\end{array}$$

commutes, where k_α and $k_{\alpha\rho^{-1}}$ are the sum inclusions. With the functoriality of this assignment clear by definition, we have thus defined the object map of $\circledast : \mathsf{S}(\widehat{\mathsf{S}C}) \to \widehat{\mathsf{S}C}$. We proceed to check that $\circledast$ is functorial. Let $(\rho,(u_j)_j) : (F_j)_{j\in\underline{n}} \to (G_j)_{j\in\underline{n}}$ be a morphism in $\mathsf{S}(\widehat{\mathsf{S}C})$. For any function $\alpha : \underline{m} \to \underline{n}$, $(x_i)_{i\in\underline{m}}$ in $\mathsf{S}C$, and $j \in \underline{n}$, one has the function $(u_j)_{(x_i)_{\alpha i=j}} : F_j(x_i)_{\alpha i=j} \to G_{\rho j}(x_i)_{\alpha i=j}$. Thus the components of $\circledast(\rho,(u_j)_j)$ are defined by the commutativity of the squares

$$\begin{array}{ccc}
\prod_{j\in\underline{n}} F_j(x_i)_{\alpha i=j} & \xrightarrow{k_\alpha} & (\underset{j\in\underline{n}}{\circledast} F_j)(x_i)_{i\in\underline{m}} \\
{\scriptstyle \prod_j(u_j)_{(x_i)_i}}\downarrow & & \downarrow{\scriptstyle \circledast(\rho,(u_j)_j)_{(x_i)_i}} \\
\prod_{j\in\underline{n}} G_{\rho j}(x_i)_{\alpha i=j} & \xrightarrow[k_{\rho\alpha}]{} & (\underset{j\in\underline{n}}{\circledast} G_j)(x_i)_{i\in\underline{m}}
\end{array}$$

for all α and $(x_i)_i$. With the functoriality of this assignment also clear by definition, we have thus defined the functor $\circledast : \mathsf{S}(\widehat{\mathsf{S}C}) \to \widehat{\mathsf{S}C}$.

3.5. Lemma. *For any small category C, the functor $\circledast : \mathsf{S}(\widehat{\mathsf{S}C}) \to \widehat{\mathsf{S}C}$ just defined describes the tensor product for Day convolution on $\mathsf{S}C$.*

Proof. The formula (4) is clearly colimit preserving in each F_j, and so it suffices to exhibit an isomorphism

$$\begin{array}{ccc}
\mathsf{S}^2C & \xrightarrow{\mathsf{S}y_{\mathsf{S}C}} & \mathsf{S}(\widehat{\mathsf{S}C}) \\
{\scriptstyle \mu_C}\downarrow & \cong & \downarrow{\scriptstyle \circledast} \\
\mathsf{S}C & \xrightarrow[y_{\mathsf{S}C}]{} & \widehat{\mathsf{S}C}
\end{array}$$

because then, this isomorphism will exhibit $\circledast$ as a pointwise left Kan extension of $y_{\mathsf{S}C}\mu_C$ along $\mathsf{S}y_{\mathsf{S}C}$, giving the result by Proposition 2.8. In terms of the notation of Section 2, this isomorphism will then be the natural isomorphism $\overline{y}_{\mathsf{S}C}$ corresponding to our formula (4) for $\circledast$.

An object of S^2C is a sequence of sequences from C, which for convenience we identify as a pair $(\psi,(c_i)_{i\in\underline{m}})$, where $\psi:\underline{m}\to\underline{n}$ is monotone, and $(c_i)_i\in\mathsf{S}C$. Applying $y_{\mathsf{S}C}\mu_C$ to this gives the representable $\mathsf{S}C(-,(c_i)_{i\in\underline{m}})$. On the other hand, for $(x_i)_{i\in\underline{l}}\in\mathsf{S}C$, the set

$$(\ast\circ\mathsf{S}(y_{\mathsf{S}C}))(\psi,(c_i)_{i\in\underline{m}})(x_i)_{i\in\underline{l}} \tag{5}$$

is, as with $\mathsf{S}C((x_i)_{i\in\underline{l}},(c_i)_{i\in\underline{m}})$, the empty set when $l\neq m$. However when $l=m$, the set (5) is the sum

$$\sum_{\alpha:\underline{m}\to\underline{n}}\prod_{j\in\underline{n}}\mathsf{S}C((x_i)_{\alpha i=j},(c_i)_{\psi i=j}). \tag{6}$$

To give an element of (6) is to give a function $\alpha:\underline{m}\to\underline{n}$, a permutation $\sigma\in\Sigma_m$ such that $\psi\sigma=\alpha$ (which just says that σ restricts to bijections $\sigma_j:\alpha^{-1}(j)\to\psi^{-1}(j)$), and for $i\in\underline{m}$ an arrow $f_i:x_i\to c_{\sigma i}$ of C. This is the same as to give a morphism $(\sigma,(f_i)_i):(x_i)_{i\in\underline{m}}\to(c_i)_{i\in\underline{m}}$ of $\mathsf{S}C$ such that $\psi\sigma=\alpha$, and this last equation shows that α is redundant. We thus have our desired isomorphism, whose naturality is very easy to check. ∎

3.6. Hereditary condition maps as the components of a natural transformation. We now return to the situation of a general symmetric colax monoidal functor $\tau:\mathsf{S}C\to M$. Denote by θ^τ the coherence 2-cell datum

$$\begin{array}{ccc} \mathsf{S}M & \xrightarrow{\ \otimes\ } & M \\ {\scriptstyle \mathsf{S}(M(\tau,1))}\downarrow & \overset{\theta^\tau}{\Longrightarrow} & \downarrow{\scriptstyle M(\tau,1)} \\ \mathsf{S}(\widehat{\mathsf{S}C}) & \xrightarrow[\ \ast\ ]{} & \widehat{\mathsf{S}C} \end{array}$$

for the symmetric lax monoidal functor $M(\tau,1):M\to\widehat{\mathsf{S}C}$. By Theorem 2.11, exactness of τ is equivalent to the invertibility of θ^τ.

Using the explicit description of the Day convolution tensor product in $\mathsf{S}(\widehat{\mathsf{S}C})$, just established in Lemma 3.5, we see that the components of θ^τ at $(w_j)_{j\in\underline{n}}$ in $\mathsf{S}M$ amount to maps of sets

$$(\theta^\tau_{(w_j)_j})_{(x_i)_i}:\sum_{\alpha:\underline{m}\to\underline{n}}\prod_{j\in\underline{n}}M(\tau(x_i)_{\alpha i=j},w_j)\longrightarrow M(\tau(x_i)_i,\bigotimes_j w_j)$$

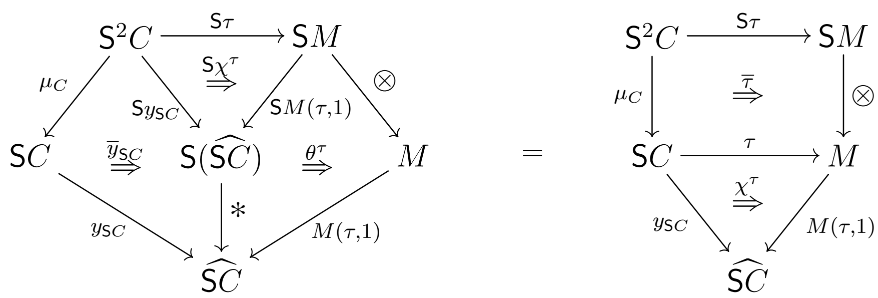
for $(x_i)_{i\in\underline{m}}$ in $\mathsf{S}C$, which we proceed to describe.

3.7. Lemma. *The component $(\theta^\tau_{(w_j)_j})_{(x_i)_i}$ is the function which sends $(\alpha,(g_j)_{j\in\underline{n}})$ to the composite*

$$\tau(x_i)_{i\in\underline{m}}\xrightarrow{\tau\sigma_\alpha}\tau(a_{\sigma_\alpha^{-1}i})_{i\in\underline{m}}\xrightarrow{\overline{\tau}}\bigotimes\nolimits_{j\in\underline{n}}\tau(x_i)_{\alpha i=j}\xrightarrow{\bigotimes_j g_j}\bigotimes\nolimits_{j\in\underline{n}}\tau(w_j) \tag{7}$$

(Note that in the special case where $w_j=\tau y_j$ (that is, the components of θ^τ in the image of τ), we recover precisely the hereditary maps $\mathbf{h}_{\tau,x,y}$.)

Proof. We saw in Section 2 that $M(\tau, 1)$ is the pointwise left Kan extension of $y_{\mathsf{S}C}$ along τ. It then follows from Theorem 2.4.4(1) of [38] that θ^τ can be characterised as the unique natural transformation satisfying the equation



(8)

To prove the lemma it is therefore enough to check that the stated formula for θ^τ satisfies Equation (8). As in the proof of Lemma 3.5 we denote an object of S^2C as a pair $(\psi, (c_i)_i)$, where $\psi : \underline{m} \to \underline{n}$ is monotone and $(c_i)_{i\in\underline{m}}$ is an object of $\mathsf{S}C$. We check that the $(\psi, (c_i)_i)$-components of either side of (8) agree. Note that for $(x_i)_{i\in\underline{l}}$ in $\mathsf{S}C$, the $((x_i)_{i\in\underline{l}}, (\psi, (c_i)_i))$-components of both sides of (8) give functions

$$\mathsf{S}C((x_i)_{i\in\underline{l}}, (c_i)_{i\in\underline{m}}) \longrightarrow M(\tau(x_i)_{i\in\underline{l}}, \textstyle\bigotimes_j \tau(c_i)_{\psi i=j}).$$

The domain of these functions is empty when $l \neq m$, and so it suffices to consider the case when $l = m$. Let $(\rho, (f_i)_{i\in\underline{m}})$ be in $\mathsf{S}C((x_i)_{i\in\underline{m}}, (c_i)_{i\in\underline{m}})$. Note that for $j \in \underline{n}$, the permutation ρ restricts to bijections $\rho_j : (\psi\rho)^{-1}(j) \to (\psi)^{-1}(j)$, and so one has $(\rho_j, (f_i)_{\psi i=j})$ in $\mathsf{S}C((x_i)_{\psi\rho i=j}, (c_i)_{\psi i=j})$. Applying the $((x_i)_i, (\psi, (c_i)_i))$-component of the left hand side of (8) to $(\rho, (f_i)_{i\in\underline{m}})$ gives the composite (7) with $g_j = \tau(\rho_j, (f_i)_{\psi i=j})$. On the other hand, applying the $((x_i)_i, (\psi, (c_i)_i))$-component of the right hand side to $(\rho, (f_i)_{i\in\underline{m}})$ gives the composite $\overline{\tau}_{(\psi,(c_i)_i)} \circ \tau(\rho, (f_i)_i)$. These coincide by the naturality of $\overline{\tau}$. ∎

Proof of Proposition 3.3 (1). In view of Lemma 3.7, the hereditary condition for τ says that the natural transformation $\theta^\tau S(\tau\eta_C)$ is invertible. The result follows since by Theorem 2.11, exactness is equivalent to the invertibility of θ^τ. ∎

The argument just given is not immediately adaptable to prove the converse. Even if τ is essentially surjective, the functor $S(\tau\eta_C) : \mathsf{S}C \to \mathsf{S}M$ is not, and so even for essentially surjective τ, it is not apparent that the invertibility of $\theta^\tau S(\tau\eta_C)$ implies the invertibility of θ^τ. Hence the need for the further assumption on τ of strong monoidality among the hypotheses of Proposition 3.3(2). Moreover, it seems to us that the clearest proof of this result follows from the consideration of factorisation categories, to which we now turn.

3.8. Factorisation category of a morphism $f : Fv \to \otimes_k w_k$. Consider a symmetric colax monoidal functor $F : V \to W$ (with coherences $\overline{F} : (\otimes_i v_i) \to \otimes_i F v_i$). For each morphism in W of the form $f : Fv \to \otimes_k w_k$, we define the category of factorisations

$\mathrm{Fact}(f)$ as having objects diagrams of the form

$$\begin{array}{ccccc} Fv & & \xrightarrow{\quad f \quad} & & \otimes_k w_k. \\ & {\scriptstyle Fh}\searrow & & \nearrow{\scriptstyle \otimes_k g_k} & \\ & F(\otimes_k u_k) & \longrightarrow & \otimes_k F(u_k) & \end{array}$$

Morphisms in $\mathrm{Fact}(f)$ are connecting maps (in V) between the u_k and the u'_k making a left-hand triangle commute already in V, and making the right hand triangle commute in W. (The middle square commutes by naturality of the coherence data.)

By Proposition 4.5.5(1) of [38], F is exact as a symmetric colax monoidal functor (= colax morphism of S-algebras) if and only if it is exact as a colax monoidal functor (= colax morphism of M-algebras). Applying Lemma 2.4 to the lax square containing F's colax M-morphism datum, gives the following explicit characterisation of exact symmetric colax monoidal functors.

3.9. Lemma. *A symmetric colax monoidal functor $F : V \to W$ as above is exact if and only if for every $f : v \to \otimes_k w_k$ the category* $\mathrm{Fact}(f)$ *is connected.* ∎

When V is $\mathsf{S}C$, and F is strict monoidal $\tau : \mathsf{S}C \to M$, the description of $\mathrm{Fact}(f)$ simplifies considerably. In this case $v = (x_i)_{i \in \underline{m}}$, and each u_k is a sequence $(z_i)_{i \in \underline{m}_k}$. Furthermore, since h is a map in $\mathsf{S}C$ it is a labelled permutation, so in particular the concatenated sequence $((z_i)_{i \in \underline{m}_k})_{k \in \underline{r}}$ is of the same length as $\underline{m}$, so altogether we can write a factorisation of $f : \tau(x_i)_{i \in \underline{m}} \to \otimes_{k \in \underline{r}} w_k$ as

$$\tau(x_i)_{i \in \underline{m}} \xrightarrow{\tau(h)} \tau((z_i)_{\beta i = k})_{k \in \underline{r}} \;=\; \otimes_{k \in \underline{r}} \otimes_{\beta i = k} \tau z_i \xrightarrow{\otimes_{k \in \underline{r}} g_k} \otimes_{k \in \underline{r}} w_k,$$

where $\beta : \underline{m} \to \underline{r}$ is monotone. Since h is a labelled permutation, we can keep the pure permutation part as a first factor, and then absorb the second factor (the 'labels') into the g_k on the right. Renaming those g_k accordingly, we arrive at a factorisation

$$\tau(x_i)_{i \in \underline{m}} \xrightarrow{\sim} \tau((x_i)_{\beta i = k})_{k \in \underline{r}} \;=\; \otimes_{k \in \underline{r}} \otimes_{\beta i = k} \tau x_i \xrightarrow{\otimes_{k \in \underline{r}} g_k} \otimes_{k \in \underline{r}} w_k.$$

Factorisations of this form we call *normalised.* Clearly every factorisation receives a morphism from its normalisation.

Proof of Proposition 3.3 (2). By strictification, we can assume that τ is strict monoidal and identity on objects. We fix $f : \tau(x_i)_{i \in \underline{m}} = \otimes_{i \in \underline{m}} \tau x_i \to \otimes_{k \in \underline{r}} w_k$ and aim to show that $\mathrm{Fact}(f)$ is connected. The identity-on-object condition means that each w_k is a tensor product of certain τy_j, say $w_k = \otimes_{\gamma j = k} \tau y_j$, where $\gamma : \underline{n} \to \underline{r}$ is monotone. Altogether,

$$\otimes_{k \in \underline{r}} w_k = \otimes_{k \in \underline{r}} \otimes_{\gamma j = k} \tau y_j = \otimes_{j \in \underline{n}} \tau y_j.$$

The map $f : \otimes_{i \in \underline{m}} \tau x_i \to \otimes_{k \in \underline{r}} w_k$ is now of the form

$$f : \otimes_{i \in \underline{m}} \tau x_i \to \otimes_{j \in \underline{n}} \tau y_j,$$

and by the hereditary condition we get a factorisation as

$$\otimes_{i\in\underline{m}}\tau x_i \xrightarrow{\sim} \otimes_{j\in\underline{n}} \otimes_{\alpha i=j} \tau x_i \xrightarrow{\otimes_{j\in\underline{n}} s_j} \otimes_{j\in\underline{n}}\tau y_j,$$

which can also be written as

$$\otimes_{i\in\underline{m}}\tau x_i \xrightarrow{\sim} \otimes_{k\in\underline{r}} \otimes_{\gamma j=k} \otimes_{\alpha i=j}\tau x_i \xrightarrow{\otimes_{k\in\underline{r}}(\otimes_{\gamma j=k} s_j)} \otimes_{k\in\underline{r}}(\otimes_{\gamma j=k}\tau y_j). \tag{9}$$

This we refer to as the standard factorisation of f. It is seen to be an object in $\mathrm{Fact}(f)$ by putting $g_k = \otimes_{\gamma j=k} s_j$. In particular we have now shown that $\mathrm{Fact}(f)$ is not empty.

Given now any other normalised object in $\mathrm{Fact}(f)$

$$\otimes_{i\in\underline{m}}\tau x_i \xrightarrow{\sim} \otimes_{k\in\underline{r}} \otimes_{\beta i=k} \tau x_i \xrightarrow{\otimes_{k\in\underline{r}} g'_k} \otimes_{k\in\underline{r}} w_k, \tag{10}$$

where $\beta : \underline{m} \to \underline{r}$ monotone, we would like to connect it to the standard factorisation just constructed. To this end we apply the hereditary condition to each of the maps

$$g'_k : \otimes_{\beta i=k}\tau x_i \to w_k = \otimes_{\gamma j=k}\tau y_j.$$

This gives us a function $\alpha_k : \beta^{-1}(k) \to \gamma^{-1}(k)$, and all these functions assemble into a function $\alpha : \underline{m} \to \underline{n}$ such that $\gamma \circ \alpha = \beta$. With reference to these α_k, the hereditary condition gives us a normalised factorisation of g_k as

$$\otimes_{\beta i=k}\tau x_i \xrightarrow{\sim} \otimes_{\gamma j=k} \otimes_{\alpha i=j} \tau x_i \xrightarrow{\otimes_{\gamma j=k} s'_j} \otimes_{\gamma j=k}\tau y_j$$

so that altogether f factors as

$$\otimes_{i\in\underline{m}}\tau x_i \xrightarrow{\sim} \otimes_{k\in\underline{r}} \otimes_{\beta i=k} \tau x_i \xrightarrow{\sim} \otimes_{k\in\underline{r}} \otimes_{\gamma j=k} \otimes_{\alpha i=j}\tau x_i \xrightarrow{\otimes_{k\in\underline{r}}\otimes_{\gamma j=k} s'_k} \otimes_{k\in\underline{r}} \otimes_{\gamma j=k} \tau y_j.$$

If we take the middle permutation to belong to the left-hand factor, then we obtain a factorisation of the standard shape (9), and by the uniqueness property in the hereditary condition, this must actually be *equal* to the standard factorisation (9), that is $s'_j = s_j$ for all $j \in \underline{n}$. On the other hand, if we let the middle permutation belong to the right-hand factor, we get precisely the given normalised factorisation (10). Hence the given normalised factorisation is connected to the standard factorisation. Since we already remarked that any factorisation is connected to its normalisation, we have altogether shown that $\mathrm{Fact}(f)$ is connected. ∎

An alternative proof can be derived from the results in Section 5: by Proposition 5.12, we may assume that τ is of the form $\mathsf{F}(\phi)$, where $\phi : C \to P$ is a substitude. It can be checked directly that for any morphism of operads ϕ, the symmetric monoidal functor $\mathsf{F}(\phi)$ is exact. See [38] (Corollary 3.4.1) for a proof.

4. Kaufmann–Ward comma-category condition

4.1. Feynman categories. Kaufmann and Ward [25] define a *Feynman category* to be a symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$ satisfying the three conditions

(1) C is a groupoid
(2) τ induces an equivalence of groupoids $\mathsf{S}C \xrightarrow{\sim} M_{\text{iso}}$
(3) τ induces an equivalence of groupoids $\mathsf{S}(M{\downarrow}C)_{\text{iso}} \xrightarrow{\sim} (M{\downarrow}M)_{\text{iso}}$.

In this section we show that the comma-category condition (3), can be reformulated in terms of the hereditary map

$$\mathbf{h} := \mathbf{h}_{\tau,x,y} : \sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} M(\textstyle\bigotimes_{\alpha i=j} \tau x_i, \tau y_j) \longrightarrow M(\textstyle\bigotimes_{i\in\underline{m}} \tau x_i, \bigotimes_{j\in\underline{n}} \tau y_j)$$

from 3.2. This is more or less implicit in [25]. This section does not seem to generalise to the enriched setting.

4.2. Proposition. *For an essentially surjective symmetric strong monoidal functor $\tau : \mathsf{S}C \to M$, the following are equivalent:*

(1) *$\tau : \mathsf{S}C \to M$ is hereditary and $\tau_{\text{iso}} : \mathsf{S}C_{\text{iso}} \to M_{\text{iso}}$ is an equivalence of groupoids*
(2) *The natural map $\tau_{\text{comma}} : \mathsf{S}(M{\downarrow}C)_{\text{iso}} \to (M{\downarrow}M)_{\text{iso}}$ is an equivalence of groupoids.*

In view of Proposition 3.3, this shows:

4.3. Corollary. *A Feynman category is precisely a regular pattern $\tau : \mathsf{S}C \to M$ for which C is a groupoid and $\tau_{\text{iso}} : \mathsf{S}C_{\text{iso}} \to M_{\text{iso}}$ is an equivalence of groupoids.* ■

Proof of Proposition 4.2. We immediately reduce to the strict situation, where τ is identity-on-objects. Lemma 4.4 below says that if just τ_{comma} is fully faithful, then also τ_{iso} is fully faithful (and therefore actually an isomorphism). So we can separate that out as a global assumption. Now τ_{comma} is full if and only if $\mathbf{h}$ is injective by Lemma 4.6, and τ_{comma} is essentially surjective if and only if $\mathbf{h}$ is surjective by Lemma 4.5. Finally, it is actually automatic that τ_{comma} is faithful (again by Lemma 4.4). ■

4.4. Lemma. *If $\tau_{\text{comma}} : \mathsf{S}(M{\downarrow}C)_{\text{iso}} \to (M{\downarrow}M)_{\text{iso}}$ is full, respectively faithful, then $\tau_{\text{iso}} : \mathsf{S}C_{\text{iso}} \to M_{\text{iso}}$ is full, respectively faithful. Conversely, if τ_{iso} is faithful then τ_{comma} is faithful.*

Proof. The first statements follow immediately from the commutative diagram

$$\begin{array}{ccc} \mathsf{S}C_{\text{iso}} & \xrightarrow{\tau_{\text{iso}}} & M_{\text{iso}} \\ \downarrow & & \downarrow \\ \mathsf{S}(M{\downarrow}C)_{\text{iso}} & \xrightarrow[\tau_{\text{comma}}]{} & (M{\downarrow}M)_{\text{iso}} \end{array}$$

since the vertical maps are fully faithful.

For the last statement, the isomorphisms in the image of τ_{comma} are of the form

$$\begin{array}{ccc} \bigotimes_j \otimes_i x_i & \xrightarrow{a} & \bigotimes_j \otimes_i x'_i \\ {\scriptstyle \otimes_j g_j}\downarrow & & \downarrow{\scriptstyle \otimes_j g'_j} \\ \otimes_j y_j & \xrightarrow[b]{} & \otimes_j y'_j. \end{array}$$

If individually the horizontal isos can arise from $\mathsf{S}C$ in at most one way, then taken together it is even harder to arise from $\mathsf{S}(M{\downarrow}C)$. ■

4.5. Lemma. *For an identity-on-objects symmetric strict monoidal functor $\tau : \mathsf{S}C \to M$, such that $\mathsf{S}C_{\text{iso}} \simeq M_{\text{iso}}$, the following are equivalent:*

(1) *The map* $\mathbf{h}$ *in the hereditary condition is surjective.*
(2) *The natural map $\tau_{\text{comma}} : \mathsf{S}(M{\downarrow}C)_{\text{iso}} \to (M{\downarrow}M)_{\text{iso}}$ is essentially surjective.*

Proof. Throughout the proof we suppress τ on objects, since anyway τ is identity-on-objects.

We first prove that if $\mathbf{h}$ is surjective, then τ_{comma} is essentially surjective. Given some object in $M{\downarrow}M$, that's precisely an element in the codomain of $\mathbf{h}$, say $f : \otimes_i x_i \to \otimes_j y_j$. By surjectivity of $\mathbf{h}$, there is an element $(\alpha, g_1, \ldots, g_n)$ in the domain of $\mathbf{h}$, whose tensor product is f. Now just the sequence $(g_1, \ldots, g_n)$ is an object in $\mathsf{S}(M{\downarrow}C)$ and by assumption, their tensor product is isomorphic to f (by the permutation σ_α obtained from α).

Conversely, assuming τ_{comma} is essentially surjective, let us prove that $\mathbf{h}$ is surjective. Given $f : \otimes_i x_i \to \otimes_j y_j$, an element in the codomain of $\mathbf{h}$, we need to construct an element on the left—that's a tuple $(\alpha, g_1, \ldots, g_n)$—such that the composite

$$\otimes_i x_i \xrightarrow{\sigma_\alpha} \otimes_j \otimes_k x_k \xrightarrow{\otimes_j g_j} \otimes_j y_j$$

is equal to f. (Here σ_α is the permutation part of α, obtained by permutation-monotone factorisation). Since τ_{comma} is essentially surjective, there exists an object $(\lambda', h'_1, \ldots, h'_n)$ in $\mathsf{S}(M{\downarrow}C)$ (here $\lambda' : \underline{m} \to \underline{n}$ is monotone, and $h'_j : \otimes_{\lambda' i' = j'} x'_{i'} \to y'_{j'}$) whose tensor product is isomorphic to f:

$$\begin{array}{ccc} \otimes_i x_i & \xrightarrow[\sim]{a} & \otimes_{j'} \otimes_{\lambda' i' = j'} x'_{i'} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \otimes_{j'} h'_{j'}} \\ \otimes_j y_j & \xrightarrow[b]{\sim} & \otimes_{j'} y'_{j'}. \end{array}$$

Since τ_{iso} is full, both a and b come from $\mathsf{S}C$, and in particular can be written as a permutation followed by a tensor product of isos in C. Let ς be the permutation underlying a and let ρ be the permutation underlying b. Put

$$\alpha := \rho^{-1} \circ \lambda' \circ \varsigma.$$

The permutation $\rho : \underline{n} \xrightarrow{\sim} \underline{n}$ is such that with $j' = \rho j$ we have $y_j \simeq y'_{j'}$. Conjugating with this isomorphism we find

$$\begin{array}{ccc} \bigotimes_j \bigotimes_{\lambda' i' = \rho j} x'_{i'} & \xrightarrow{\tilde{b}} & \bigotimes_{j'} \bigotimes_{\lambda' i' = j'} x'_{i'} \\ {\scriptstyle \otimes_j g'_j} \downarrow & & \downarrow {\scriptstyle \otimes_j h'_{j'}} \\ \bigotimes_j y_j & \xrightarrow[b]{} & \bigotimes_{j'} y'_{j'}. \end{array}$$

(The permutation $\tilde{\rho} : \underline{m} \xrightarrow{\sim} \underline{m}$ underlying $\tilde{b}$ permutes the blocks according to the permutation b.) Except for some isos in C, the new map g'_j is essentially $h'_{\rho j}$. The tensor product of the new maps g'_j now have underlying indexing map $\lambda := \rho^{-1} \circ \lambda' \circ \tilde{\rho}$, which is different from λ', but is still monotone. On the other hand, the permutation $\varsigma : \underline{m} \xrightarrow{\sim} \underline{m}$ is such that with $i' = \varsigma i$ we have $x_i \simeq x'_{i'}$. Using this, we can rewrite the upper left-hand corner

$$\bigotimes_j \bigotimes_{\lambda i' = \rho j} x'_{i'} \simeq \bigotimes_j \bigotimes_{\lambda \varsigma i = \rho j} x_i = \bigotimes_j \bigotimes_{\alpha i = j} x_i,$$

but a little care is needed with this substitution, since it may permute stuff inside each j-factor. However, this permutation can be absorbed into each g'_j and now called g_j (and this does not affect λ), giving altogether

$$\begin{array}{ccccc} \bigotimes_i x_i & \xrightarrow{\sigma} & \bigotimes_j \bigotimes_{\alpha i = j} x_i & \longrightarrow & \bigotimes_{j'} \bigotimes_{\lambda' i' = j'} x'_{i'} \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \otimes_j g_j} & & \downarrow {\scriptstyle \otimes_{j'} h'_{j'}} \\ \bigotimes_j y_j & \xrightarrow[=]{} & \bigotimes_j y_j & \longrightarrow & \bigotimes_j y'_{j'}. \end{array}$$

The remaining permutation σ is monotone on λ-fibres by construction, and since $\alpha = \lambda \circ \sigma$, we see that $(\alpha, g_1, \ldots, g_n)$ is a solution to our problem: by construction, $\mathbf{h}$ applied to $(\alpha, g_1, \ldots, g_n)$ is the original f. Hence $\mathbf{h}$ is surjective. ∎

4.6. Lemma. *For an identity-on-objects symmetric strict monoidal functor $\tau : \mathsf{S}C \to M$, such that $\mathsf{S}C_{\text{iso}} \simeq M_{\text{iso}}$, the following are equivalent:*

1. *The map $\mathbf{h}$ in the hereditary condition is injective.*
2. *The natural map $\tau_{\text{comma}} : \mathsf{S}(M{\downarrow}C)_{\text{iso}} \to (M{\downarrow}M)_{\text{iso}}$ is full.*

Proof. Throughout the proof we suppress τ on objects, since anyway τ is identity-on-objects.

Let us show that if τ_{comma} is full then $\mathbf{h}$ is injective. Suppose we have two elements in the domain of $\mathbf{h}$ both giving f. Say (α, g_j) and (α', g'_j). (In both cases j runs to the same n: that's part of the data in f). This gives us now a commutative diagram of maps in M:

$$\begin{array}{ccccc} \bigotimes_j \otimes_i x_i & \xleftarrow{\sigma} & \otimes_i x_i & \xrightarrow{\sigma'} & \bigotimes_j \otimes_i x_i \\ {\scriptstyle \otimes_j g_j} \downarrow & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle \otimes_j g'_j} \\ \otimes_j y_j & = & \otimes_j y_j & = & \otimes_j y_j \end{array}$$

The outer vertical maps, $\otimes_j g_j$ and $\otimes_j g'_j$, are now two objects in $M{\downarrow}M$, exhibited isomorphic by means of $\sigma' \circ \sigma^{-1}$ and the identity at the bottom. Since τ_{comma} is full, this isomorphism comes from one in $\mathsf{S}(M{\downarrow}C)$, hence is given by a labelled permutation of $\underline{n}$. Since the bottom map is the identity, also the permutation $\sigma' \circ \sigma^{-1}$ is the identity on the outer tensor factors, those indexed by j. So σ and σ' agree on outer factors. But they are also monotone on fibres, so in fact they must agree completely. It follows that $g_j = g'_j$, and hence in particular also that $\alpha = \alpha'$.

Conversely, assuming that $\mathbf{h}$ is injective, let us show that τ_{comma} is full. Given two objects in $M{\downarrow}M$ in the image of $\mathsf{S}(M{\downarrow}C)$, pictured vertically, and an iso between them (consisting of two isomorphisms, pictured horizontally):

$$\begin{array}{ccc} \bigotimes_j \otimes_i x_i & \xrightarrow{a} & \bigotimes_j \otimes_i x'_i \\ {\scriptstyle \otimes_j g_j}\big\downarrow & & \big\downarrow{\scriptstyle \otimes_j g'_j} \\ \otimes_j y_j & \xrightarrow[b]{} & \otimes_j y'_j \end{array}$$

A priori, we don't know that the j run to the same n, but in fact they do, because of the existence of b: since τ_{iso} is fully faithful, b is in fact the image of an isomorphism in $\mathsf{S}C$, which is to say that it is a labelled permutation. For the same reason, a is a labelled permutation too. Since the vertical maps are in the image of τ_{comma}, their corresponding α and α' have trivial permutation part. It follows that the permutation a must actually be a refinement of the permutation b. In particular we have necessarily $\alpha = \alpha'$, which is a monotone map, since it just comes from the tensor product.

It remains to check that a and b together actually form a valid isomorphism in $\mathsf{S}(M{\downarrow}C)$. We have found that the original square is the tensor product of a sequence of squares of the form

$$\begin{array}{ccc} \otimes_{i\in\alpha^{-1}\sigma^{-1}j} x_i & \xrightarrow{a} & \otimes_{i\in\alpha^{-1}(j)} x'_i \\ {\scriptstyle g_j}\big\downarrow & & \big\downarrow{\scriptstyle g'_j} \\ y_{\sigma^{-1}j} & \xrightarrow[b]{} & y'_j \end{array}$$

It remains to check that each of them commutes. For fixed $j \in \underline{n}$, the two composites in this individual square are both elements in the set

$$M\Big(\bigotimes_{i\in\alpha^{-1}\sigma^{-1}(j)} x_i,\ y'_j\Big),$$

so altogether they form a tuple which is an element in

$$\prod_{j\in\underline{n}} M\Big(\bigotimes_{i\in\alpha^{-1}\sigma^{-1}(j)} x_i,\ y'_j\Big).$$

That's in the $\sigma \circ \alpha$ summand of the domain of $\mathbf{h}$. If one of the squares did not commute, it would thus constitute two distinct elements in this set, with the same image under tensoring (the map $\mathbf{h}$). Since $\mathbf{h}$ is injective, we conclude that in fact all those squares do commute, and hence form a valid isomorphism in $\mathsf{S}(M{\downarrow}C)$, as required. ∎

5. Hermida-type adjunctions and the main theorem

5.1. Operads. By *operad* we always mean symmetric coloured operad (in $\boldsymbol{Set}$), also known as symmetric multicategory. Hence, an operad P consists of a set I of objects (also called colours), for each pair $(x_1, ..., x_n; y)$ consisting of a finite sequence of *input objects* $(x_1, ..., x_n)$ and one *output object* y, a set $P(x_1, ..., x_n; y)$ of operations from $(x_1, \ldots, x_n)$ to y. Moreover there is an identity operation $1_x : (x) \to x$ in $P(x; x)$, and a substitution law satisfying the usual axioms.

We shall use various shorthand notation for sequences $(x_1, \ldots, x_n)$, such as $(x_i)_{i\in\underline{n}}$, or just $(x_i)_i$, or even $\mathbf{x}$, when practical.

Operads form a 2-category $\boldsymbol{Opd}$, the morphisms being morphisms of operads in the usual sense. A 2-cell

$$P \underset{g}{\overset{f}{\rightrightarrows}} Q \qquad (\Downarrow \omega)$$

consists of components $\omega_x : (fx) \to gx$, required to be natural with respect to all operations of P, that is, given an operation $b : (x_i)_i \to y$, one has $\omega_y f(b) = g(b)(\omega_{x_i})_i$.

A small category can be regarded as an operad with only unary operations, and in this way $\boldsymbol{Cat}$ becomes a coreflective sub-2-category of $\boldsymbol{Opd}$ (the coreflector picks out the unary part of an operad.) Various notions from category theory makes sense also for operads: in particular, a morphism of operads $f : P \to Q$ is called *fully faithful* if it induces bijections on multihomsets $P(\mathbf{x}; y) \xrightarrow{\sim} Q(f\mathbf{x}; f(y))$. Gabriel factorisation works the same for operads as for categories, as exploited also in [18]: every operad morphism factors as bijective-on-objects followed by fully faithful. Just as in the category case, this is an enhanced factorisation system. And just as in the category case, one can always choose the bijective-on-objects part to be actually identity-on-objects (with which choice the factorisation is unique).

5.2. Hermida adjunctions. For any symmetric monoidal category M, its *endomorphism operad* $\mathrm{End}(M)$ has objects those of M, and sets of operations given by

$$\mathrm{End}(M)((x_1, ..., x_n); y) \quad = \quad M(\textstyle\bigotimes_{i=1}^{n} x_i, y).$$

The category P-$\mathrm{Alg}(M)$, of P-algebras in M, is defined as

$$P\text{-}\mathrm{Alg}(M) := \boldsymbol{Opd}(P, \mathrm{End}(M))$$

which at the level of objects, says that a P-algebra in M is a morphism of operads $P \to \mathrm{End}(M)$. The assignment $M \mapsto \mathrm{End}(M)$ is the effect on objects of a 2-functor $\mathrm{End} : \boldsymbol{SMC} \to \boldsymbol{Opd}$, where $\boldsymbol{SMC}$ denotes the 2-category of symmetric monoidal categories, symmetric strong monoidal functors, and monoidal natural transformations. Below, we will also consider the restriction of this 2-functor to a 2-functor $\boldsymbol{SMC}_{\mathrm{s}} \to \boldsymbol{Opd}$, where $\boldsymbol{SMC}_{\mathrm{s}}$ is the locally full sub-2-category of $\boldsymbol{SMC}$ consisting of the symmetric strict monoidal categories and symmetric strict monoidal functors.

In the other direction, one can associate to any operad P, a symmetric strict monoidal category $\mathsf{F}P$, which has the following explicit description. An object of $\mathsf{F}P$ is a finite sequence of objects of P. A morphism $(x_1, ..., x_m) \to (y_1, ..., y_n)$ in $\mathsf{F}P$, consists of an *indexing function* $\alpha : \underline{m} \to \underline{n}$, together with for each $j \in \underline{n}$, an operation $b_j : (x_i)_{\alpha i=j} \to y_j$. The category structure of $\mathsf{F}P$ comes from substitution in P and the composition of (indexing) functions. Thus the homs of $\mathsf{F}P$ are given by

$$\mathsf{F}P\big((x_i)_{i\in\underline{m}}, (y_j)_{j\in\underline{n}}\big) = \sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} P((x_i)_{\alpha i=j}; y_j).$$

Note that for a category C regarded as an operad, we have a canonical identification $\mathsf{F}C = \mathsf{S}C$.

Now, $\mathsf{F}P$ enjoys a strict universal property amongst all symmetric strict monoidal categories, expressed by the isomorphisms of categories on the left

$$\boldsymbol{SMC}_{\mathrm{s}}(\mathsf{F}P, M) \cong \boldsymbol{Opd}(P, \mathrm{End}(M)) \qquad \boldsymbol{SMC}(\mathsf{F}P, M) \simeq \boldsymbol{Opd}(P, \mathrm{End}(M))$$

2-naturally in M; and also a bicategorical universal property amongst all symmetric monoidal categories expressed by the equivalences of categories on the right, which are pseudo-natural in M. Taken together one thus has a 2-adjunction as indicated on the left

$$\boldsymbol{SMC}_{\mathrm{s}} \underset{\mathrm{End}}{\overset{\mathsf{F}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \boldsymbol{Opd} \qquad\qquad \boldsymbol{SMC} \underset{\mathrm{End}}{\overset{\mathsf{F}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp_{\mathrm{b}}}}}} \boldsymbol{Opd}$$

and a biadjunction as indicated on the right. We shall call these the *Hermida adjunctions* in honour of Claudio Hermida, who studied the strict version in the non-symmetric case [22] (see also [16], Theorem 4.2). In [37] Corollary 6.4.7, they were obtained formally from the 2-monad S.

Recall that to give a biadjunction with left adjoint F and right adjoint End is to give pseudo-natural transformations $\eta^H : 1_{\boldsymbol{Opd}} \to \mathrm{End}\,\mathsf{F}$ and $\varepsilon^H : \mathsf{F}\,\mathrm{End} \to 1_{\boldsymbol{SMC}}$ together with invertible modifications

$$\begin{array}{ccc} \mathsf{F} & \xrightarrow{\mathsf{F}\eta^H} & \mathsf{F}\,\mathrm{End}\,\mathsf{F} \\ & {\scriptstyle 1_{\mathsf{F}}}\searrow \ \cong \ \swarrow{\scriptstyle \varepsilon^H\mathsf{F}} & \\ & \mathsf{F} & \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{End} & \xrightarrow{\eta^H\,\mathrm{End}} & \mathrm{End}\,\mathsf{F}\,\mathrm{End} \\ & {\scriptstyle 1_{\mathrm{End}}}\searrow \ \cong \ \swarrow{\scriptstyle \mathrm{End}\,\varepsilon^H} & \\ & \mathrm{End} & \end{array}$$

sometimes called the left and right *triangulators* for the biadjunction. In our case the 2-adjunction can be recovered from the biadjunction by restricting from $\boldsymbol{SMC}$ to $\boldsymbol{SMC}_{\mathrm{s}}$. In terms of the unit and counit, this says the following: (1) the unit of the Hermida biadjunction is the same as that of the 2-adjunction and so is strictly 2-natural; (2) the left triangulator is an identity; (3) for morphisms of $\boldsymbol{SMC}_{\mathrm{s}}$ the associated ε^H pseudo-naturality datum is an identity; and (4) for objects of $\boldsymbol{SMC}_{\mathrm{s}}$ the associated component of the right triangulator is an identity. We turn now to an explicit description of the components of η^H and ε^H.

The counit component at $M \in \boldsymbol{SMC}$

$$\varepsilon^H_M : \mathsf{F}(\mathrm{End}(M)) \longrightarrow M \qquad (x_i)_i \mapsto \bigotimes_i x_i$$

has object map as indicated, and hom functions

$$\mathsf{F}(\mathrm{End}(M))\big((x_i)_{i\in\underline{m}}, (y_j)_{j\in\underline{n}}\big) \longrightarrow M\big(\bigotimes_{i\in\underline{m}} x_i, \bigotimes_{j\in\underline{n}} y_j\big)$$

which send

$$(\alpha : \underline{m} \to \underline{n}, (g_j : \bigotimes_{\alpha i = j} x_i \to y_j)_j)$$

to the composite

$$\bigotimes_{i\in\underline{m}} x_i \xrightarrow{\sigma} \bigotimes_{i\in\underline{m}} x_{\sigma^{-1}i} \cong \bigotimes_{j\in\underline{n}} \bigotimes_{\alpha i=j} x_i \xrightarrow{\otimes_j g_j} \bigotimes_{j\in\underline{n}} y_j$$

where $\sigma = \sigma_\alpha$ is the permutation part of α (given by the permutation/monotone factorisation), and the unnamed isomorphism is obtained from the coherences for M, these being identities when M is strict.

The component of the unit at $P \in \boldsymbol{Opd}$ is given on objects as

$$\begin{aligned} \eta^H_P : P &\longrightarrow \mathrm{End}(\mathsf{F}P) \\ x &\longmapsto (x). \end{aligned}$$

More interesting is to see what η^H does on sets of operations: it is a map

$$P(x_1, \dots, x_m; y) \to \mathrm{End}(\mathsf{F}P)\big((x_1), \dots, (x_m); (y)\big)$$

but the last set we can unravel as

$$= \mathsf{F}P\big((x_1, \dots, x_m), (y)\big)$$

since the tensor product in $\mathsf{F}P$ is just concatenation of sequences;

$$= \sum_{\alpha:\underline{m}\to\underline{1}} \prod_{j\in\underline{1}} P((x_i)_{\alpha i=j}; y_j)$$

by definition of hom sets in $\mathsf{F}P$;

$$= P(x_1, \dots, x_m; y)$$

since there exists only one indexing map $\underline{m} \to \underline{1}$. In the end, η^H_P is the identity map on operations. In conclusion:

5.3. Lemma. *The components of η^H, the unit for the Hermida adjunction, are fully faithful operad maps.* ∎

5.4. Pinned operads. We use the term *pinned* for an object (e.g. a symmetric monoidal category, an operad, or a substitude) equipped with a map singling out some objects (the pins).

A *pinned operad* is a triple (C, ϕ, P), in which C is a category regarded as an operad with only unary operations, P is an operad, and $\phi : C \to P$ is a morphism of operads. Pinned operads assemble into a 2-category $\boldsymbol{pOpd}$. A morphism $(C_1, \phi_1, P_1) \to (C_2, \phi_2, P_2)$ is a triple (f, g, ω) consisting of a functor $f : C_1 \to C_2$, an operad morphism $g : P_1 \to P_2$, and an invertible 2-cell $\phi_2 f \to g\phi_1$ in $\boldsymbol{Opd}$. A 2-cell $(f, g, \omega) \to (f', g', \omega')$ is a pair (α, β), where $\alpha : f \to f'$ is a natural transformation, and $\beta : g \to g'$ is a 2-cell of $\boldsymbol{Opd}$, such that

$$\begin{array}{ccc} C_1 & \xrightarrow{\phi_1} & P_1 \\ f \Big\downarrow \overset{\alpha}{\Rightarrow} \Big\downarrow f' & \overset{\omega'}{\Rightarrow} & \Big\downarrow g' \\ C_2 & \xrightarrow[\phi_2]{} & P_2 \end{array} \quad = \quad \begin{array}{ccc} C_1 & \xrightarrow{\phi_1} & P_1 \\ f \Big\downarrow & \overset{\omega}{\Rightarrow} & g \Big\downarrow \overset{\beta}{\Rightarrow} \Big\downarrow g' \\ C_2 & \xrightarrow[\phi_2]{} & P_2 \end{array}$$

in $\boldsymbol{Opd}$. Compositions for $\boldsymbol{pOpd}$ are inherited from $\boldsymbol{Opd}$ in the obvious way. A morphism (f, g, ω) of $\boldsymbol{pOpd}$ is said to be *strict* when ω is an identity 2-cell, and the wide and locally full sub-2-category of $\boldsymbol{pOpd}$ consisting of the strict morphisms is denoted $\boldsymbol{pOpd}_{\mathrm{s}}$. The 2-categories $\boldsymbol{pSMC}$ and $\boldsymbol{pSMC}_{\mathrm{s}}$ of pinned symmetric monoidal categories, and pinned symmetric strict monoidal categories, were described above in 1.4.

5.5. Pinned Hermida adjunctions. The Hermida adjunctions have pinned analogues

$$\boldsymbol{pSMC}_{\mathrm{s}} \underset{\mathrm{End}_{\mathrm{p}}}{\overset{\mathsf{F}_{\mathrm{p}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \boldsymbol{pOpd}_{\mathrm{s}} \qquad\qquad \boldsymbol{pSMC} \underset{\mathrm{End}_{\mathrm{p}}}{\overset{\mathsf{F}_{\mathrm{p}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot_{\mathrm{b}}}}}} \boldsymbol{pOpd}$$

which we now describe. The object maps of $\mathrm{End}_{\mathrm{p}}$ and F_{p} are

$$\mathsf{S}C \xrightarrow{\tau} M \quad \longmapsto \quad C \xrightarrow{\eta^H_C} \mathrm{End}(\mathsf{S}C) \xrightarrow{\mathrm{End}(\tau)} \mathrm{End}(M) \qquad\qquad C \xrightarrow{\phi} P \quad \longmapsto \quad \mathsf{S}C \xrightarrow{\mathsf{F}(\phi)} \mathsf{F}P$$

respectively, and since End and F are 2-functors and η^H is 2-natural, these definitions extend in the obvious way to arrows and 2-cells.

The component of the unit η^{p} of $\mathsf{F}_{\mathrm{p}} \dashv_b \mathrm{End}_{\mathrm{p}}$ at (C, ϕ, P) is $(1_C, \eta^H_P, \mathrm{id})$, as depicted in the diagram on the left

$$\begin{array}{ccccc} C & & \xrightarrow{\phi} & & P \\ \Big\| & & & & \Big\downarrow \eta^H_P \\ C & \xrightarrow[\eta^H_C]{} & \mathrm{End}(\mathsf{S}C) & \xrightarrow[\mathrm{End}(\mathsf{F}(\phi))]{} & \mathrm{End}(\mathsf{F}P) \end{array} \qquad \begin{array}{ccccc} \mathsf{S}C & \xrightarrow{\mathsf{F}(\eta^H_C)} & \mathsf{F}(\mathrm{End}(\mathsf{S}C)) & \xrightarrow{\mathsf{F}(\mathrm{End}(\tau))} & \mathsf{F}(\mathrm{End}(M)) \\ \Big\| & \swarrow \varepsilon^H_{\mathsf{S}C} & & \overset{\varepsilon^H_\tau}{\cong} & \Big\downarrow \varepsilon^H_M \\ \mathsf{S}C & & \xrightarrow[\tau]{} & & M \end{array}$$

which commutes by the naturality of η^H. So the components of η^{p} all live in $\boldsymbol{pOpd}_{\mathrm{s}}$. The component of the counit ε^{p} of $\mathsf{F}_{\mathrm{p}} \dashv_b \mathrm{End}_{\mathrm{p}}$ at (C, τ, M) is $(1_C, \varepsilon^H_M, \varepsilon^H_\tau \mathsf{F}(\eta^H_C))$, as depicted on the right in the previous display. Note that when (C, τ, M) is strict, the pseudo-naturality datum ε^H_τ is an identity, and then this component of ε^{p} lives in $\boldsymbol{pOpd}_{\mathrm{s}}$. The pseudo-naturality data for the unit and counit, and the left and right triangulators for $\mathsf{F}_{\mathrm{p}} \dashv_b \mathrm{End}_{\mathrm{p}}$ are inherited from $\mathsf{F} \dashv_b \mathrm{End}$. The manner in which the biadjunction restricts to a 2-adjunction is clearly inherited also.

5.6. SUBSTITUDES [14]. The notion of substitude was introduced by Day and Street [14] as a general setting for substitution. It is a common generalisation of operad and (symmetric) monoidal category. See the appendix of [5] for a concise account of the basic theory. A substitude is like an operad, but allowing for a category of objects instead of just a set of objects. The data is

– a category C
– a functor $P : (\mathsf{S}C)^{\mathrm{op}} \times C \to \boldsymbol{Set}$
– composition and unit laws, subject to non-surprising axioms.

Substitudes can be described also as monads in the bicategory of generalised species [17]. For the present purposes, the most convenient is to package the definition into the following (cf. [15, 6.3] for the equivalence between the two formulations of the definition): A *substitude* is a pinned operad $\phi : C \to P$ in which ϕ is the identity on objects. We denote by $\boldsymbol{Subst}$ the full sub-2-category of $\boldsymbol{pOpd}$ consisting of the substitudes, and denote by $\mathsf{J} : \boldsymbol{Subst} \to \boldsymbol{pOpd}$ the inclusion.

5.7. SUBSTITUDE COREFLECTION. Since the bijective-on-objects morphisms form the left class of an enhanced factorisation system (Gabriel factorisation), by general principles, the inclusion functor $\mathsf{J} : \boldsymbol{Subst} \to \boldsymbol{pOpd}$ has a right adjoint, denoted $(-)'$, forming a 2-adjunction

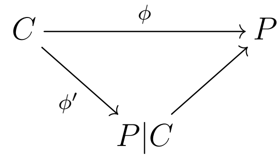
$$\boldsymbol{pOpd} \underset{(-)'}{\overset{\mathsf{J}}{\underset{\perp}{\leftrightarrows}}} \boldsymbol{Subst}.$$

Explicitly, given a pinned operad (C, ϕ, P), factoring ϕ as identity-on-objects then fully faithful as on the left in

$$\begin{array}{ccc} C & \xrightarrow{\quad\phi\quad} & P \\ & {\scriptstyle\phi'}\searrow \quad \nearrow{\scriptstyle\varepsilon_\phi} & \\ & P' & \end{array} \qquad\qquad \begin{array}{ccccc} C_1 & \xrightarrow{\phi'_1} & P'_1 & \xrightarrow{\varepsilon_{\phi_1}} & P_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & \overset{\omega'}{\cong} & \downarrow{\scriptstyle g} \\ C_2 & \xrightarrow[\phi'_2]{} & P'_2 & \xrightarrow[\varepsilon_{\phi_2}]{} & P_2 \end{array}$$

one obtains the substitude (C, ϕ', P') together with the (C, ϕ, P)-component of the counit of $\mathsf{J} \dashv (-)'$, which we denote as ε_ϕ. Given a morphism $(f, g, \omega) : (C_1, \phi_1, P_1) \to (C_2, \phi_2, P_2)$ of $\boldsymbol{pOpd}$, one induces f' and ω' uniquely so that the composite on the right in the previous display is ω, using the enhancedness of $\boldsymbol{Opd}$'s Gabriel factorisation. Using the fact that this Gabriel factorisation is also $\boldsymbol{Cat}$-enriched, one can easily exhibit $(-)'$'s 2-cell mapping explicitly.

It is suggestive to write the operad part of $\phi' : C \to P'$ as $P|C$: it is the operad base-changed to its pins.



The operad $P|C$ has the same objects as C by construction, and operations

$$P|C(x_1, \ldots, x_m; y) = P(\phi x_1, \ldots, \phi x_m; \phi y).$$

5.8. The substitude biadjunction. Now compose the pinned Hermida biadjunction 5.5 with the coreflection of substitudes into pinned operads 5.7:

$$\boldsymbol{pSMC} \underset{\mathsf{F}_{\mathrm{p}}}{\overset{\mathrm{End}_{\mathrm{p}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp_b}}}} \boldsymbol{pOpd} \underset{(-)'}{\overset{\mathsf{J}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \boldsymbol{Subst}$$

The composed biadjunction

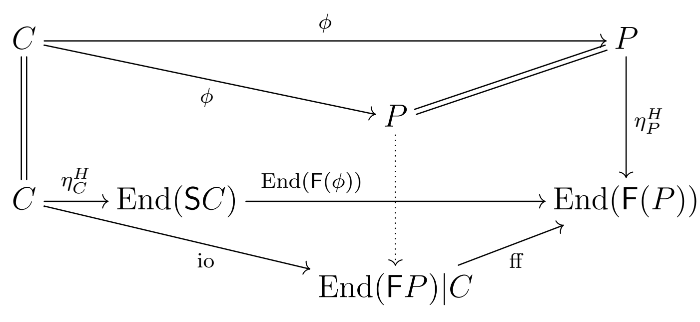
$$\boldsymbol{pSMC} \underset{\mathrm{End}_{\mathrm{iop}}}{\overset{\mathsf{F}_{\mathrm{iop}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp_b}}}} \boldsymbol{Subst} \tag{11}$$

goes like this: the left adjoint takes a substitude $C \to P$ to the pinned symmetric monoidal category $\mathsf{S}C \to \mathsf{F}P$, and the right adjoint takes a pinned symmetric monoidal category $\mathsf{S}C \to M$ to $C \to \mathrm{End}(M)|C$.

Unlike the Hermida biadjunction and the pinned version, this one has invertible unit:

5.9. Proposition. *The unit for the substitude biadjunction* $\mathsf{F}_{\mathrm{iop}} \dashv \mathrm{End}_{\mathrm{iop}}$ *is invertible.*

Proof. Let $\phi : C \to P$ be a substitude. In the diagram



the back rectangle is the unit η^{p}_{ϕ}. The unit we are after, $\eta^{\mathrm{iop}}_{\phi}$, is obtained by Gabriel factorising its horizontal arrows: this induces the dotted arrow by functoriality of the factorisation, and the left-hand square is now $\eta^{\mathrm{iop}}_{\phi}$. It is clearly invertible if and only if the dotted arrow is invertible. But the dotted arrow is invertible because it compares two Gabriel factorisations of $\mathrm{End}(\mathsf{F}(\phi)) \circ \eta^H_C$: on one hand ϕ followed by η^H_P (the latter being fully faithful by Lemma 5.3) and on the other hand the factorisation at the bottom of the diagram. ∎

5.10. Counit of the substitude biadjunction. Thanks to Proposition 5.9 the 2-category ***Subst*** is biequivalent to the full sub-2-category of ***pSMC***, spanned by those pinned symmetric monoidal categories (C, τ, M) for which the the counit $\varepsilon_\tau^{\mathrm{iop}}$ is an equivalence. In the remainder of this section, we determine for which (C, τ, M) this is the case. Fundamental to this characterisation is the hereditary condition 3.2. We begin the discussion by characterising equivalences in ***pSMC***.

5.11. Lemma. *A morphism $(f, g, \omega) : (C_1, \tau_1, M_1) \to (C_2, \tau_2, M_2)$ of **pSMC** is an equivalence if and only if the functors f and g are equivalences of categories.*

Proof. It suffices to show that the forgetful 2-functor

$$\boldsymbol{pSMC} \longrightarrow \boldsymbol{Cat} \times \boldsymbol{Cat} \qquad (C, \tau, M) \mapsto (C, M)$$

reflects equivalences. So we suppose that $(f, g, \omega) : (C_1, \tau_1, M_1) \to (C_2, \tau_2, M_2)$ is a morphism of ***pSMC*** such that f and g are equivalences of categories. Choose adjoint pseudo inverses f' and g' of f and g respectively, and then note that g' admits a unique symmetric strong monoidal structure making the adjoint equivalence it participates in in ***Cat***, into one in ***SMC***. Thus we have adjoint equivalences $(\mathsf{S}f, \mathsf{S}f')$ and (g, g') in ***SMC***, and so one can take the mate $\omega' : \tau_1 \mathsf{S}(f') \cong g'\tau_2$ of ω. This makes (f', g', ω') into a morphism of ***pSMC***. Since ω and ω' are mates via the adjoint equivalences $(\mathsf{S}f, \mathsf{S}f')$ and (g, g'), these assemble to an adjoint equivalence in ***pSMC***, exhibiting (f', g', ω') as an adjoint pseudo-inverse of (f, g, ω). ∎

We now unpack the counit component $\varepsilon_\tau^{\mathrm{iop}}$ for (C, τ, M) in ***pSMC***. First we describe the substitude $\mathrm{End}_{\mathrm{iop}}(\tau)$. It is obtained by Gabriel factorising the pinned operad $\mathrm{End}_{\mathrm{p}}(\tau)$, which is the top composite here:

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C^H} & \mathrm{End}(\mathsf{S}C) & \xrightarrow{\mathrm{End}(\tau)} & \mathrm{End}(M) \\ & {\scriptstyle \mathrm{End}_{\mathrm{iop}}(\tau)} \searrow & & \nearrow {\scriptstyle \mathrm{ff}}\,{\scriptstyle \epsilon} & \\ & & \mathrm{End}(M)|C & & \end{array}$$

Note that the operad $\mathrm{End}(M)|C$ has objects those of C, and operation sets

$$\mathrm{End}(M)|C\big(x_1, \ldots, x_m; y\big) = \mathrm{End}(M)\big(\tau x_1, \ldots, \tau x_m; \tau y\big) = M\big(\tau x_1 \otimes \cdots \otimes \tau x_m, \tau y\big).$$

Now apply $\mathsf{F}_{\mathrm{iop}}$, rendered as the top square in the next diagram; the diagram as a whole is the counit $\varepsilon_\tau^{\mathrm{iop}}$, the key part being of course the right-hand vertical composite:

$$\begin{array}{ccccc} \mathsf{S}C & & \xrightarrow{\mathsf{F}\,\mathrm{End}_{\mathrm{iop}}(\tau)} & & \mathsf{F}(\mathrm{End}(M)|C) \\ \| & & & & \downarrow {\scriptstyle \mathsf{F}(\epsilon)} \\ \mathsf{S}C & \xrightarrow{\mathsf{F}\eta_{\mathsf{S}C}^H} & \mathsf{F}\,\mathrm{End}(\mathsf{S}C) & \xrightarrow{\mathsf{F}\,\mathrm{End}(\tau)} & \mathsf{F}\,\mathrm{End}(M) \\ \| & \swarrow {\scriptstyle \varepsilon_{\mathsf{S}C}^H} & & \overset{\varepsilon_\tau^H}{\cong} & \downarrow {\scriptstyle \varepsilon_M^H} \\ \mathsf{S}C & & \xrightarrow[\tau]{} & & M \end{array} \tag{12}$$

5.12. PROPOSITION. *For a pinned symmetric monoidal category $\tau : \mathsf{S}C \to M$, the counit component $\varepsilon_\tau^{\mathrm{iop}}$ is an equivalence if and only if τ is essentially surjective and satisfies the hereditary condition 3.2.*

PROOF. By Lemma 5.11, $\varepsilon_\tau^{\mathrm{iop}}$ is an equivalence in $\boldsymbol{pSMC}$ if and only if $\varepsilon_M^H\mathsf{F}(\epsilon)$ is an equivalence of categories. Since $\mathsf{F}\,\mathrm{End}_{\mathrm{iop}}(\tau)$ is an identity on objects, Diagram (12) shows that $\varepsilon_M^H\mathsf{F}(\epsilon)$ is essentially surjective if and only if τ is. So we have reduced to the case where τ is essentially surjective, which is the content of the next lemma. ■

5.13. LEMMA. *An essentially surjective pinned symmetric monoidal category $\tau : \mathsf{S}C \to M$ satisfies the hereditary condition if and only if $\varepsilon_M^H\mathsf{F}(\epsilon)$ is fully faithful.*

PROOF. Applying Power coherence, we can reduce to the case where τ is an identity-on-objects symmetric strict monoidal functor. Let us now unpack the effect on morphisms of $\varepsilon_M^H\mathsf{F}(\epsilon)$ in this case. The objects of the symmetric monoidal category $\mathsf{F}(\mathrm{End}(M)|C)$ are sequences of objects in C, and $\varepsilon_M^H\mathsf{F}(\epsilon)$ sends $(x_1,\dots,x_m)$ to $\tau x_1 \otimes \cdots \otimes \tau x_m$. The hom sets of $\mathsf{F}(\mathrm{End}(M)|C)$ are

$$\mathsf{F}(\mathrm{End}(M)|C)\big((x_i)_{i\in\underline{m}},(y_j)_{j\in\underline{n}}\big) = \sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} M\Big(\bigotimes_{\alpha i=j}\tau x_i, \tau y_j\Big),$$

and the hom mapping of $\varepsilon_M^H\mathsf{F}(\epsilon)$ into $M(\otimes_i\tau x_i, \otimes_j \tau y_j)$, sends $(\alpha, g_1,\dots,g_n)$ to the composite

$$\bigotimes_{i\in\underline{m}}\tau x_i \xrightarrow{\ \sigma\ } \bigotimes_{i\in\underline{m}}\tau x_{\sigma^{-1}i} = \bigotimes_{j\in\underline{n}}\bigotimes_{\alpha i=j}\tau x_i \xrightarrow{\otimes_j g_j} \bigotimes_{j\in\underline{n}}\tau y_j.$$

Thus, the hom functions for $\varepsilon_M^H\mathsf{F}(\epsilon)$ are exactly the functions $\mathbf{h}_{\tau,x,y}$ whose bijectivity is the hereditary condition. ■

Taking Theorem 2.11 and Propositions 3.3 and 5.12 together, we have thus arrived at the first part of the main theorem:

5.14. THEOREM. $\mathsf{F}_{\mathrm{iop}} : \boldsymbol{Subst} \to \boldsymbol{pSMC}$ *induces a biequivalence* $\boldsymbol{Subst} \sim \boldsymbol{RPat}$. ■

5.15. OPERADS. There are three ways of regarding operads as substitudes. For a given operad P, the options are: the trivial pinning, where $C = \mathrm{obj}(P)$, the discrete category of objects; the canonical groupoid pinning, where $C = P_1^{\mathrm{iso}}$, the groupoid of invertible unary operations; and the full pinning, where $C = P_1$, the category of all unary operations.

We are interested at the moment in the canonical groupoid pinning, $P_1^{\mathrm{iso}} \to P$. Note that the functor $(-)^{\mathrm{iso}} : \boldsymbol{Cat} \to \boldsymbol{Grpd}$ is *not* a 2-functor. However it does make sense to apply $(-)^{\mathrm{iso}}$ to *invertible* natural transformations, and so $(-)^{\mathrm{iso}}$ is $\boldsymbol{Grpd}$-enriched. If, for any 2-category $\mathcal{K}$, we denote by $(\mathcal{K})^{2\text{-iso}}$ its underlying groupoid-enriched category, obtained from $\mathcal{K}$ simply by ignoring non-invertible 2-cells, then another way to express these considerations is to say that one has a 2-functor $(-)^{\mathrm{iso}} : (\boldsymbol{Cat})^{2\text{-iso}} \to \boldsymbol{Grpd}$.

Thus, the process of taking the canonical groupoid pinning is the effect on objects of a 2-functor

$$\mathsf{G} : (\boldsymbol{Opd})^{2\text{-iso}} \longrightarrow \boldsymbol{Subst}$$

which is 2-fully-faithful when the codomain is restricted to $(\boldsymbol{Subst})^{\text{2-iso}}$. Writing $\boldsymbol{FeynCat}$ for the full sub-2-category of $(\boldsymbol{pSMC})^{\text{2-iso}}$ consisting of the Feynman categories in the sense of Kaufmann and Ward, the second part of our main theorem is as follows.

5.16. THEOREM. $\mathsf{F}_{\text{iop}} \circ \mathsf{G}$ *restricts to a biequivalence* $(\boldsymbol{Opd})^{\text{2-iso}} \sim \boldsymbol{FeynCat}$.

PROOF. First we check that a pinned symmetric monoidal category in the image is a Feynman category, using 4.3. We already know that it satisfies the hereditary condition, and by construction P_1^{iso} is a groupoid. It remains to check that

$$\mathsf{S}P_1^{\text{iso}} \to (\mathsf{F}P)_{\text{iso}}$$

is an equivalence. These two categories have the same objects, namely sequences of objects in P. The arrows in $\mathsf{F}P$ from $\mathbf{x}$ to $\mathbf{y}$ form the hom set

$$\sum_{\alpha:\underline{m}\to\underline{n}} \prod_{j\in\underline{n}} P((x_i)_{i\in\alpha^{-1}(j)}; y_j)$$

They are composed as operations in P. There is an obvious forgetful functor to $\boldsymbol{Set}$, given by returning the indexing set for the sequence. For an arrow to be invertible, at least its underlying α must be a bijection, and furthermore it is then clear that the involved operations have to be invertible unary operations.

Conversely, if we start with a Feynman category $\tau : \mathsf{S}C \to M$, the image in $\boldsymbol{Subst}$ is the substitude $C \to \text{End}(M)|C$, and since C is a groupoid and $\mathsf{S}C \simeq M_{\text{iso}}$, it is clear that the invertible unary operations of $\text{End}(M)|C$ are precisely those of C, which is the condition for the substitude to be in the image of G. ∎

5.17. TWO OTHER SUBCATEGORIES OF $\boldsymbol{RPat}$ EQUIVALENT TO THE CATEGORY OF OPERADS. Let us note that there are two other subcategories of $\boldsymbol{RPat}$ which are equivalent to the category of operads, given by the two other natural embeddings of $\boldsymbol{Opd}$ into $\boldsymbol{Subst}$: one takes an operad P to its discrete pinning $\text{obj}(P) \to P$—this is a 1-functor only, as it is not defined on 2-cells. The other embedding takes P to its full pinning $P_1 \to P$ (this is a honest 2-functor).

The first corresponds to the subcategory of discrete substitudes, which in turn corresponds to the subcategory of regular patterns $\mathsf{S}C \to M$ for which C is discrete. While this is obviously a stronger condition than the first Feynman-category condition, that of being a groupoid, on the other hand the second Feynman-category condition, $\mathsf{S}C \xrightarrow{\sim} M_{\text{iso}}$ is not in general satisfied. Imposing this second condition on top of the discreteness condition gives the notion of discrete Feynman category, mentioned in [25] to be related to operads. More precisely this notion corresponds to operads in which there are no unary operations other than the identities (the locus of operads for which the groupoid-pinning and discrete-pinning embeddings coincide). (This embedding of the 1-category of operads into regular patterns was also observed by Getzler [19].)

The second embedding of operads into substitudes, endowing an operad P with its full pinning $P_1 \to P$, has as essential image that of *normal* substitudes, namely those whose unary operations are precisely those coming from C. These can also be characterised as those for which the square constituting the unique substitude morphism to the terminal substitude $1 \to \mathrm{Comm}$ is a pullback:

$$\begin{array}{ccc} C & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \mathrm{Comm}. \end{array}$$

The corresponding sub-2-category in $\boldsymbol{RPat}$ is the 2-full sub-2-category spanned by the regular patterns $\mathsf{S}C \to M$ which are pullbacks of the terminal regular pattern $\mathsf{S}1 \to \boldsymbol{FinSet}$. In detail, for any (strictified) regular pattern, which in virtue of Theorem 5.14 is of the form $\mathsf{S}C \to \mathsf{F}P$, we have a commutative square of symmetric strong monoidal functors

$$\begin{array}{ccc} \mathsf{S}C & \longrightarrow & \mathsf{F}P \\ \downarrow & & \downarrow \\ \mathsf{S}1 & \longrightarrow & \boldsymbol{FinSet}, \end{array}$$

which clearly is a pullback precisely when the previous square is.

The normality condition plays an important role in the theory of generalised multicategories of Cruttwell and Shulman [12], a more general framework which includes substitudes as a special case. Their generalised multicategories are monoids in T-spans, for T a monad on a virtual equipment (a particular kind of double category). They are called normalised when the unary operations coincide with the vertical arrows in the double category, and the importance of the condition is that non-normalised generalised multicategories can be reinterpreted as normalised ones, by adjusting the monad and the ambient virtual equipment. Cruttwell and Shulman also consider the discrete-objects case, but do not consider the intermediate invertible case, which in the present situation is the most interesting.

5.18. Algebras for substitudes. Given a substitude $\phi : C \to P$, and a symmetric monoidal category W an algebra consists of

- a functor $A : C \to W$
- for each $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in C$, an action map

$$P(\mathbf{x}; y) \longrightarrow W(A(x_1) \otimes \cdots \otimes A(x_n), A(y))$$

satisfying some conditions. One of these conditions says that for every arrow $f : x \to y$ in C, the action of the unary operation $\phi(f)$ is equal to $A(f) : A(x) \to A(y)$. The remaining conditions are those of an algebra for the operad P; in fact, the first condition implies that functoriality in C follows from the P-algebra axioms, so to give an algebra for $\phi : C \to P$ in W amounts just to give an algebra for the operad P in W.

Just as in the case of operads, the notion of algebra can also be described in terms of a substitude of endomorphisms: the *endomorphism substitude* E^A of a functor $A : C \to W$

is given in elementary terms as the substitude with category C and $E^A(x_1, \ldots, x_n; y) = W(A(x_1) \otimes \cdots \otimes A(x_n), A(y))$. An algebra structure on $A : C \to W$ is now the same thing as an identity-on-C substitude map $\phi \to E^A$, that is, an operad map $P \to E^A$ under C. The endomorphism substitude can be described more conceptually as

$$C \to \mathrm{End}(W)|C,$$

so altogether the notion of algebra is conveniently formulated in terms of the substitude adjunction: the endomorphism substitude of $A : C \to W$ is precisely $\mathrm{End}_{\mathrm{iop}}(\tau^A)$ where $\tau^A : \mathsf{S}C \to W$ is the tautological factorisation of A through $\mathsf{S}C$, and an algebra is an identity-on-C substitude map $\phi \to \mathrm{End}_{\mathrm{iop}}(\tau^A)$. By adjunction, this is the same thing as an identity-on-C pinned symmetric strong monoidal functor

$$\mathsf{F}_{\mathrm{iop}}(\phi) \to \tau^A.$$

Giving this amounts just to giving $\alpha : \mathsf{F}P \to W$ (a symmetric strong monoidal functor), that is, an algebra for the regular pattern corresponding to ϕ.

This works also for algebra homomorphisms: a homomorphism of substitude algebras is a natural transformation $u : A \Rightarrow B$ compatible with the action maps. In terms of endomorphism substitudes, this compatibility amounts to commutativity of the diagram (of operads under C)

$$\begin{array}{ccccc} & & E^A & & \\ & \nearrow & & \searrow^{u(-,*)} & \\ P & & & & E^{A,B} \\ & \searrow & & \nearrow_{u(*,-)} & \\ & & E^B & & \end{array} \tag{13}$$

where $E^{A,B}$ is the W-bimodule $\mathsf{S}C^{\mathrm{op}} \times C \to W$ given by $E^{A,B}(\mathbf{x}; y) = W(A(x_1) \otimes \cdots \otimes A(x_n), B(y))$, $u(-,*)$ is induced by postcomposition with u_y, and $u(*,-)$ is induced by precomposition with $u_{x_1} \otimes \cdots \otimes u_{x_n}$. Now the natural transformation $u : A \Rightarrow B$ extends tautologically to a monoidal natural transformation $\tau^u : \tau^A \Rightarrow \tau^B$, which in turn extends to a monoidal natural transformation

$$\mathsf{F}P \underset{\beta}{\overset{\alpha}{\underset{\Downarrow}{\rightrightarrows}}} W,$$

which is the corresponding homomorphism of regular-pattern algebras. The components of this monoidal natural transformation are the same as those of τ^u, since $\mathsf{S}C$ and $\mathsf{F}P$ have the same object set; naturality in arrows in $\mathsf{F}P$ is a consequence of (13), and monoidality follows from construction.

Clearly this construction can be reversed, to construct an substitude-algebra homomorphism from a monoidal natural transformation. Altogether we obtain the following proposition.

5.19. Proposition. *Algebras for a substitude are the same thing as algebras for the corresponding regular pattern. More precisely for each symmetric monoidal category W, there is an equivalence of categories between the category of algebras in W for a substitude, and the category of algebras in W for the corresponding regular pattern.* ∎

Since algebras for a substitude $\phi : C \to P$ are just algebras for the operad P, we immediately get also, via the embedding ***Opd*** → ***Subst*** by canonical groupoid pinning:

5.20. Corollary. *Algebras for an operad are the same thing as algebras for the corresponding Feynman category.* ∎

Appendix: Power coherence

In this appendix we recall coherence for symmetric monoidal categories, from the point of view of Power's general approach to coherence [33]. This point of view takes as input the 2-monad S on ***Cat*** for symmetric monoidal categories, and produces the coherence theorem for symmetric monoidal categories. The formulation of this result given in Lemma A.4 below, is most convenient for us for the purposes of studying regular patterns and Feynman categories.

A.1. 2-monads and their algebras. A 2-*monad* T on a 2-category $\mathcal{K}$ is just a monad in the sense of ***Cat***-enriched category theory. Thus one has the usual data of a monad

$$T : \mathcal{K} \longrightarrow \mathcal{K} \qquad \eta : 1_{\mathcal{K}} \longrightarrow T \qquad \mu : T^2 \longrightarrow T$$

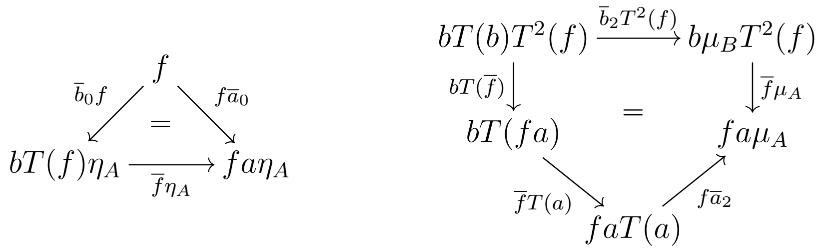
but where T is a 2-functor, and η and μ are 2-natural, and the axioms are written down exactly as before. The extra feature is that now one has several different types of algebras, and several different types of algebra morphisms, and thus a variety of alternative 2-categories of algebras. For instance a *pseudo T-algebra* structure on $A \in \mathcal{K}$ consists of the data of an action $a : TA \to A$, as well as invertible coherence 2-cells $\overline{a}_0 : 1_A \to a\eta_A$ and $\overline{a}_2 : aT(a) \to a\mu_A$, which satisfy:

$$\begin{array}{ccc} a & \xrightarrow{\overline{a}_0 a} & a\eta_A a \\ & \overset{\mathrm{id}}{\searrow} \;\; = & \downarrow{\scriptstyle \overline{a}_2\eta_{TA}} \\ & & a \end{array} \qquad \begin{array}{ccc} aT(a)T^2(a) & \xrightarrow{\overline{a}_2T^2(a)} & a\mu_AT^2(a) \\ {\scriptstyle aT(\overline{a}_2)}\downarrow & = & \downarrow{\scriptstyle \overline{a}_2\mu_{TA}} \\ aT(a)T(\mu_A) & \xrightarrow[\overline{a}_2T(\mu_A)]{} & a\mu_A\mu_{TA} \end{array} \qquad \begin{array}{ccc} aT(a)T(\eta_A) & \xleftarrow{aT(\overline{a}_0)} & a \\ {\scriptstyle \overline{a}_2T(\eta_A)}\downarrow & = \;\; \overset{\mathrm{id}}{\swarrow} & \\ a & & \end{array}$$

When these coherence isomorphisms are identities, we have a *strict* T-algebra on A. There are also algebra types in which the coherence 2-cells are non-invertible, but these are not important for us here.

A *lax morphism* $(A, a) \to (B, b)$ between pseudo T-algebras is a pair $(f, \overline{f})$, where

$f : A \to B$ and $\overline{f} : bT(f) \to fa$, satisfying the following axioms:



When $\overline{f}$ is an isomorphism, f is said to be a *pseudo morphism*, and when $\overline{f}$ is an identity, f is said to be *strict*. Given lax T-algebra morphisms f and $g : (A, a) \to (B, b)$, a T-algebra 2-cell $f \to g$ is a 2-cell $\phi : f \to g$ in $\mathcal{K}$ such that $\overline{g}(bT(\phi)) = (\phi a)\overline{f}$. In the case $T = \mathsf{S}$ one has

- lax morphism = symmetric lax monoidal functor,
- pseudo morphism = symmetric strong monoidal functor,
- strict morphism = symmetric strict monoidal functor, and
- algebra 2-cell = monoidal natural transformation.

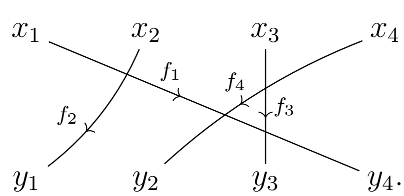
The standard notations for some of the various 2-categories of algebras of a 2-monad T are

- Ps-T-Alg: objects are pseudo T-algebras, morphisms are pseudo morphisms,
- T-$\mathrm{Alg}_{\mathrm{s}}$: objects and morphisms are strict, and
- T-$\mathrm{Alg}_{\mathrm{l}}$: objects are strict and morphisms are lax,

A.2. The free symmetric-monoidal-category 2-monad. The 2-monad S is described explicitly in Section 5.1 of [36]. For a category C, an object of $\mathsf{S}C$ is a finite sequence of objects of C, and a morphism is of the form

$$(\rho, (f_i)_{1 \leqslant i \leqslant n}) : (x_i)_{1 \leqslant i \leqslant n} \longrightarrow (y_i)_{1 \leqslant i \leqslant n}$$

where $\rho \in \Sigma_n$ is a permutation, and for $i \in \underline{n} = \{1, ..., n\}$, $f_i : x_i \to y_{\rho i}$. Intuitively such a morphism is a permutation labelled by the arrows of C, as in



The unit $\eta_C : C \to \mathsf{S}C$ is given by the inclusion of sequences of objects of length 1. The multiplication $\mu_C : \mathsf{S}^2 C \to \mathsf{S}C$ is given on objects by concatenation, and on arrows by the substitution of labelled permutations. Given a symmetric monoidal category M, $(X_i)_i \mapsto \bigotimes_i X_i$ is the effect on objects of a functor $\bigotimes : \mathsf{S}M \to M$, whose arrow map is described using the symmetries of M. This is the action of a pseudo S-algebra structure on M. The unit coherences (those denoted $\overline{a}_0$ in the general definition) in this case are identities, and the components of $\overline{a}_2$ come from the associators and unit coherences of M. The strictness of M as a symmetric monoidal category, is the same thing as its strictness as an S-algebra. In fact, aside from the strictness of the unit coherences, and the specification of the tensor product as an n-ary tensor product for all n, rather than just the cases $n = 0$ (the unit usually written as I) and $n = 2$ (the usual binary tensor product $(A, B) \mapsto A \otimes B$), a pseudo S-algebra is *exactly* a symmetric monoidal category in the usual sense.

The coherence theorem for symmetric monoidal categories says that every symmetric monoidal category is equivalent to a strict one. Mac Lane's original proof of this involved a detailed combinatorial analysis. However from the point of view of Power's general approach [33], this result comes out of how S interacts with the *Gabriel factorisation system* on $\boldsymbol{Cat}$, by which every functor $f : A \to B$ is factored as

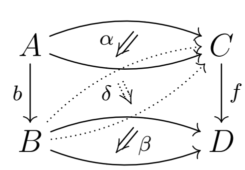
$$A \xrightarrow{g} C \xrightarrow{h} B$$

where g is bijective on objects, and h is fully faithful.

A.3. The Gabriel factorisation system. The Gabriel factorisation system, in which the left class is that of bijective-on-objects functors, and the right class consists of the fully faithful functors, is an orthogonal factorisation system, meaning that arrows in the left class admit a unique lifting property with respect to arrows in the right class. One way to formulate this, is to say that for any bijective-on-objects functor $b : A \to B$, and fully faithful functor $f : C \to D$, the square

$$\begin{array}{ccc} \boldsymbol{Cat}(B,C) & \xrightarrow{\boldsymbol{Cat}(b,C)} & \boldsymbol{Cat}(A,C) \\ {\scriptstyle \boldsymbol{Cat}(B,f)}\downarrow & & \downarrow{\scriptstyle \boldsymbol{Cat}(A,f)} \\ \boldsymbol{Cat}(B,D) & \xrightarrow[\boldsymbol{Cat}(b,D)]{} & \boldsymbol{Cat}(A,D) \end{array}$$

is a pullback in the category of sets. In fact the Gabriel factorisation is enriched over $\boldsymbol{Cat}$, meaning that these squares are also pullbacks in $\boldsymbol{Cat}$. Explicitly on arrows this says that given α and β as in



such that $f\alpha = \beta b$, there exists a unique δ as shown such that $\alpha = \delta b$ and $f\delta = \beta$. Moreover the Gabriel factorisation system is an *enhanced* factorisation system in the sense of Kelly [28], which is to say that given u, v and α as on the left,

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle b}\downarrow & \overset{\alpha}{\cong} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[v]{} & D \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle b}\downarrow & \overset{d}{\nearrow}\ \overset{\beta}{\cong} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[v]{} & D \end{array}$$

there exists unique d and β as shown on the right, such that $u = db$ and $\alpha = \beta b$. It is not difficult to verify that the Gabriel factorisation system enjoys these properties [34].

From the explicit description of S it is easy to see that S preserves bijective-on-objects functors.

With these details in hand Power's idea boils down to the following. Given a symmetric strong monoidal functor $F : S \to M$ where S is a symmetric *strict* monoidal category, one can take the Gabriel factorisation

$$S \xrightarrow{G} M' \xrightarrow{H} M$$

of F, and then factor the coherence witnessing F as strong monoidal on the left

$$\begin{array}{ccc} \mathsf{S}S & \xrightarrow{\otimes} & S \\ {\scriptstyle \mathsf{S}(F)}\downarrow & \overset{\overline{F}}{\cong} & \downarrow{\scriptstyle F} \\ \mathsf{S}M & \xrightarrow[\otimes]{} & M \end{array} \qquad = \qquad \begin{array}{ccc} \mathsf{S}S & \xrightarrow{\otimes} & S \\ {\scriptstyle \mathsf{S}(G)}\downarrow & = & \downarrow{\scriptstyle G} \\ \mathsf{S}M' & \overset{\otimes}{\dashrightarrow} & M' \\ {\scriptstyle \mathsf{S}(H)}\downarrow & \overset{\overline{H}}{\cong} & \downarrow{\scriptstyle H} \\ \mathsf{S}M & \xrightarrow[\otimes]{} & M \end{array}$$

uniquely as on the right, using enhancedness and the fact that $\mathsf{S}G$ is bijective on objects.

A.4. Lemma. [33] *In the situation just described, $\otimes : \mathsf{S}M' \to M'$ is a symmetric strict monoidal structure on M'. With respect to this structure, G is a symmetric strict monoidal functor, and $\overline{H}$ is the coherence datum making H into a symmetric strong monoidal functor.* ■

We attribute this result to Power, although it is not exactly formulated in this way in [33]. Power's result applies to more general monads T than S (only required to preserved bijective-on-objects functors), but for the functor F he only considers the case where $S = TM$ and F is the action $TM \to M$. It is easy to adjust his argument to the present situation, to verify the strict S-algebra axioms for $\otimes : \mathsf{S}M' \to M'$, and the pseudo S-morphism axioms for $\overline{H}$. Note that G is a strict S-algebra morphism by construction.

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Department of Mathematics, Macquarie University

Departament de matemàtiques, Universitat Autònoma de Barcelona

Faculty of Mathematics and Physics, Charles University, Prague

Email: michael.batanin@mq.edu.au
kock@mat.uab.cat
mark.weber.math@gmail.com